PERIOD MAPS AND THE C_{st} CONJECTURE

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Abstract. We construct a family of elliptic curves over an open disc whose total space violates the C_{st} conjecture in degree 1. Hence the conjecture, known for proper and for Stein varieties, as well as for products of such, does not extend to proper families over Stein bases.

This is an instance of a general criterion: the C_{st} conjecture holds for the total space of a smooth proper family over a sufficiently small open disc if and only if its period map is constant, i.e., all Gauss–Manin Higgs fields vanish or, equivalently, the p-adic local systems $\mathrm{R}^i f_\star \mathbf{Q}_p$ are geometrically constant. This criterion is special to curves: over a 2-dimensional polydisc, we give a family of p-adic analogues of non-Kähler elliptic surfaces whose period maps and local systems are constant, but whose total space violates the conjecture.

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1. Introduction

Let K be a complete discretely valued field of mixed characteristic $(0,p)$ with perfect residue field k, and let $C = \widehat{\overline{K}}$. Let X be a smooth, connected, dagger variety over K. Define in[1] $\mathcal{D}(\mathbf{Q}_{p,\square})$:

$$\mathrm{DR}(X) := \mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_K^{\mathrm{L}\square} \mathbf{B}_{\mathrm{dR}}^+, \quad \mathrm{DR}^i(X) := H^i(\mathrm{DR}(X)) = H^i_{\mathrm{dR}}(X) \otimes_K^{\square} \mathbf{B}_{\mathrm{dR}}^+,$$
$$F^{r,i}(X) := H^i(F^r\mathrm{DR}(X)), \quad \mathrm{HK}^{r,i}(X) := \left(H^i_{\mathrm{HK}}(X_C) \otimes_{\breve{C}}^{\square} \mathbf{B}_{\mathrm{st}}^+\right)^{N=0,\varphi=p^r},$$

where $H^i_{\mathrm{HK}}(-)$ denotes the Hyodo–Kato cohomology and $\breve{C} = \mathrm{Frac}(W(\overline{k}))$. We have the following conjecture [6, Conj. 1.4], which describes the geometric pro-étale cohomology of X in terms of de Rham data:

Conjecture 1.1. *For $r \geq i \geq 0$, the square*

$$\begin{array}{ccc} H^i_{\mathrm{proét}}(X_C, \mathbf{Q}_p(r)) & \longrightarrow & \mathrm{HK}^{r,i}(X) \\ \downarrow & & \downarrow \iota_{\mathrm{HK}} \\ F^{r,i}(X) & \longrightarrow & \mathrm{DR}^i(X) \end{array}$$

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[1] We write $\mathbf{Q}_{p,\square}$ for the category of solid $\mathbf{Q}_p$-modules.

is bicartesian in $\mathcal{D}(\mathbf{Q}_{p,\square})$. *Equivalently, we have the exact sequence*

$$0 \longrightarrow H^i_{\text{proét}}(X_C, \mathbf{Q}_p(r)) \longrightarrow \mathrm{HK}^{r,i}(X) \oplus F^{r,i}(X) \longrightarrow \mathrm{DR}^i(X) \longrightarrow 0. \tag{1.2}$$

For $i = 0$, since X is assumed to be connected, this sequence becomes $0 \to \mathbf{Q}_p(r) \to (\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^r} \oplus t^r\mathbf{B}^+_{\mathrm{dR}} \to \mathbf{B}^+_{\mathrm{dR}} \to 0$, which is one of the possible ways to write the standard fundamental exact sequence. Hence Conjecture 1.1 is true for $i = 0$, and we will assume $i \geq 1$ in the rest of the paper.

In [6], this conjecture is proved for proper, Stein, and dagger-affinoid varieties, for analytifications of algebraic varieties, for almost proper varieties, and for products of a smooth proper variety with a smooth Stein variety; see [7, Section 4] for a recent survey. In [7, Rem. 4.4], proper fibrations over Stein bases were proposed as a natural extension of the product case. In this paper we show that, without further hypotheses on the variation of the Hodge filtration, this extension fails, and that over bases of dimension ≥ 2 it can fail even when the Hodge filtration does not vary.

Assume $H^*_{\mathrm{dR}}(X)$ to be finite dimensional. If $j \leq r$, let $\mathcal{E}^{r,j}(X)$ be the vector bundle on the Fargues–Fontaine curve X_{FF} associated to the φ-module[2] $H^j_{\mathrm{HK}}(X_C)\{-r\}$ modified at x_∞ (the point defined by $t = 0$) by imposing $\widehat{\mathcal{O}}_{X_{\mathrm{FF}},x_\infty} \otimes \mathcal{E}^{r,j}(X) = F^r\mathrm{DR}^j(X)$. Then we have an exact sequence [6, Cor. 7.9]

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(X_{\mathrm{FF}}, \mathcal{E}^{r,i-1}(X)) & & & & H^1(X_{\mathrm{FF}}, \mathcal{E}^{r,i}(X)) \longrightarrow & 0 \\ & \downarrow & & & & \uparrow & \\ & H^i_{\text{proét}}(X_C, \mathbf{Q}_p(r)) & \longrightarrow & \mathrm{HK}^{r,i}(X) \oplus F^{r,i}(X) & \longrightarrow & \mathrm{DR}^i(X) & \end{array} \tag{1.3}$$

So, in order to find a counterexample to Conjecture 1.1 in degree i, it is enough to find X, such that $H^1(X_{\mathrm{FF}}, \mathcal{E}^{r,i}(X)) \neq 0$.

The idea behind the construction of such an example is very simple. Let $X \to S$ be proper and smooth and, if $s \in S$, let X_s be the fiber over s and $j_s : X_s \to X$ be the corresponding closed immersion. Assume $j_s^* : H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}}(X_s)$ is an isomorphism. Then $\mathcal{E}^{r,i}(X) \to \mathcal{E}^{r,i}(X_s)$ is a modification at x_∞. But $\mathcal{E}^{r,i}(X_s)$ is pure of slope 0 since X_s is proper; hence $H^1(X_{\mathrm{FF}}, \mathcal{E}^{r,i}(X)) = 0$ if and only if $\mathcal{E}^{r,i}(X) \to \mathcal{E}^{r,i}(X_s)$ is an isomorphism, which is equivalent to $j_s^* : F^aH^i_{\mathrm{dR}}(X) \to F^aH^i_{\mathrm{dR}}(X_s)$ being an isomorphism for all a (see Section 4.2 for details). Hence one has to find a family for which j_s^* induces an isomorphism $H^i_{\mathrm{dR}}(X) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X_s)$, but not of the filtrations.

This is achieved (Theorem 3.8) by taking a smooth proper family of elliptic curves over a dagger disc for which the H^1_{dR} of the total space can be identified with that of a fiber, while its Hodge filtration is strictly smaller than that of the fiber (in fact, it is the part common to the Hodge filtrations of all the fibers, see Remark 4.11). The same calculation applies over the open disc and gives a counterexample (Corollary 3.30) among smooth partially proper varieties that are proper over a Stein curve (a special case of holomorphically convex spaces).

The above discussion makes it possible to give a local criterion for Conjecture 1.1 to hold for a family over a small enough disc. Let $f : \mathcal{X} \to U$ be a smooth proper family over a smooth rigid analytic curve, and let $S \subset U$ be a sufficiently small dagger or open disc (see Hypothesis 4.5 below). Horizontal transport identifies the de Rham cohomologies of all the fibers X_u, $u \in S(K)$, with a fixed space V^i, and the Hodge filtrations of the fibers become a family of filtrations of V^i: this is the period map of the family. The Hodge filtration of the total space $X = \mathcal{X} \times_U S$ is the part common to all these filtrations (Remark 4.11), and X satisfies Conjecture 1.1 in degree i if and only if the period maps in degrees $i-1$ and i are constant (Corollary 4.13). Equivalently (Theorem 4.9), the Higgs fields in degrees $i-1$ and i vanish identically on S, or the p-adic local systems $\mathrm{R}^{i-1}f_*\mathbf{Q}_p$ and $\mathrm{R}^if_*\mathbf{Q}_p$ are constant on S_C. For abelian schemes, the criterion reduces to the vanishing of the Kodaira–Spencer map; for elliptic curves, to the constancy of the j-invariant on S (Corollary 4.14).

[2]The twist $\{-r\}$ corresponds to dividing the Frobenius by p^r.

This criterion is special to curves (see Section 4.3). Over a polydisc of dimension ≥ 2, the conjecture still implies that the period maps and the local systems are constant, but the converse fails: we give a family over a 2-dimensional polydisc whose fibers are p-adic analogues of non-Kähler elliptic surfaces (the same construction over $\mathbf{P}^1$ gives the Hopf surface), whose period maps and local systems are constant, but whose total space violates Conjecture 1.1 in degrees 2 and 3. The obstruction takes values in global 2-forms on the base, hence vanishes when the base is a curve. (On the étale side, global 2-forms on a polydisc S appear as $H^2_{\text{proét}}(S_C, \mathbf{Q}_p(2)) \simeq \Omega^2(S_C)$, see [2, Th. 1.8]). We do not know whether such an example exists with f projective.

Remark 1.4. Conjecture 1.6 of [6] allows one to extract de Rham and Hyodo–Kato cohomologies from pro-étale cohomology. This conjecture is actually a consequence of Conjecture 1.1 when the latter is true (see [6, Chap. 9]). In cases where Conjecture 1.1 fails, one cannot hope to recover de Rham and Hyodo–Kato cohomologies via the simple recipe given in [6, Conj. 1.6], but one must instead use the analogue of the derived formulation of [3] (used to recover de Rham and Hyodo–Kato cohomologies from *compactly supported* pro-étale cohomology). The methods developed in [4] allow one to prove the resulting statement for quasi-compact dagger varieties, but there are subtleties with extensions of condensed Fréchet spaces that prevent one from proving the result for (non-proper) partially proper spaces.

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Notation and conventions. Let p be a fixed prime, and $\mathbf{Q}_p$ the field of p-adic numbers. Let K be a complete discretely valued field of mixed characteristic $(0,p)$. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$. Let G_K be the Galois group of K. We will denote by $\mathbf{B}_{\mathrm{cr}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}$ the crystalline, semistable, and de Rham period rings of Fontaine.

All rigid analytic spaces considered will be over K or C. We assume that they are separated, taut, and countable at infinity.

2. A criterion for the conjecture to be true

The following criterion was used in various forms in [6], but was not spelled out.

Proposition 2.1. *Let $f : X \to U$ be a morphism of dagger varieties defined over K. Suppose:*

- *U satisfies Conjecture 1.1.*
- *$H^*_{\mathrm{dR}}(U)$ is finite dimensional and $f^* : H^*_{\mathrm{dR}}(U) \to H^*_{\mathrm{dR}}(X)$ is an isomorphism.*

Then X satisfies Conjecture 1.1.

Proof. If $\mathcal{E}^{r,j}(Y)$, for $Y = U, X$, is the vector bundle associated to $(H^j_{\mathrm{HK}}(Y_C)\{-r\}, F^r(\mathrm{DR}^j(Y)))$ as in the introduction, then Conjecture 1.1 for Y and $i \leq r$ is equivalent to $H^1(X_{\mathrm{FF}}, \mathcal{E}^{r,j}(Y)) = 0$ for $j = i-1, i$ (see (1.3)). Now f^* induces an exact sequence $0 \to \mathcal{E}^{r,j}(U) \to \mathcal{E}^{r,j}(X) \to Z^{r,j} \to 0$, where $Z^{r,j}$ is a coherent sheaf supported at the point x_∞, which implies $H^1(X_{\mathrm{FF}}, Z^{r,j}) = 0$. If U satisfies Conjecture 1.1, we have $H^1(X_{\mathrm{FF}}, \mathcal{E}^{r,j}(U)) = 0$, which implies that $H^1(X_{\mathrm{FF}}, \mathcal{E}^{r,j}(X)) = 0$, as wanted. □

Remark 2.2. (i) Proposition 2.1 is in fact true without the hypothesis that H^*_{dR} is finite dimensional. To see this, one has to go back to the basic comparison diagram [6, Sec. 7.2.1].

(ii) As we shall see, knowing that the conjecture is true for X and that $f^* : H^*_{\mathrm{dR}}(U) \to H^*_{\mathrm{dR}}(X)$ is an isomorphism is not enough to deduce that the conjecture is true for U. In fact, if X is proper, $\mathcal{E}^{r,j}(X)$ is of slope 0 and, in order for U to verify Conjecture 1.1 for i, we need to have $\mathcal{E}^{r,j}(U)$

also of slope 0 (otherwise some of its slopes would be < 0 and $H^1(X_{\mathrm{FF}}, \mathcal{E}^{r,j}(U))$ would be $\neq 0$), which is the case if and only if f^* induces an isomorphism of filtered modules $H^j_{\mathrm{dR}}(U) \simeq H^j_{\mathrm{dR}}(X)$, for $j = i, i-1$.

3. A counterexample

Let X be a smooth dagger variety over K. We define the naive filtration on $\mathrm{DR}^i(X)$ as

$$F^r\mathrm{DR}^i(X) := \mathrm{Im}\big(H^i(F^r\mathrm{DR}(X)) \longrightarrow H^i(\mathrm{DR}(X))\big). \tag{3.1}$$

If $H^i_{\mathrm{dR}}(X)$ is finite dimensional and $i \leq r$, one has also (see Proposition A.1)

$$F^r\mathrm{DR}^i(X) = \sum_{a+b=r} F^b H^i_{\mathrm{dR}}(X) \otimes t^a \mathbf{B}^+_{\mathrm{dR}} \tag{3.2}$$

If (1.2) is exact, the map

$$\pi^{r,i} : \mathrm{HK}^{r,i}(X) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{DR}^i(X)/F^r\mathrm{DR}^i(X) \tag{3.3}$$

must be surjective. That is, non-surjectivity of (3.3) is enough to disprove Conjecture 1.1.

Remark 3.4. The relation with the proof of Proposition 2.1 is that $\mathrm{Coker}(\pi^{r,i}) = H^1(X_{\mathrm{FF}}, \mathcal{E}^{r,i}(X))$.

3.1. **The counterexample.** Assume $p \geq 5$, and take $K = \mathbf{Q}_p$. Choose $a \in \mathbf{Z}_p$ such that the elliptic curve

$$E: \quad y^2z = x(x-z)(x-az) \tag{3.5}$$

has good ordinary reduction (in particular, $a \not\equiv 0, 1 \bmod p$; any value of a which does not give a CM elliptic curve will do for a density 1 of p's; for an explicit example, one can take $a = -3$ and $p = 5$).

Let $S = \mathrm{Sp}\big(\mathbf{Q}_p\langle T\rangle^\dagger\big), \lambda = a + p^2T$, and let $f : X \to S$ be the projective cubic

$$y^2z = x(x-z)(x-\lambda z), \tag{3.6}$$

so that the fiber at $T = 0$ is $j : E \hookrightarrow X$. The dagger structure in (3.6) is obtained by restriction from the same analytic family on a disc S' larger than the closed unit disc, for example from $v_p(T) > -1$. Since $v_p(p^2T) > 0$ on that disc, both λ and $\lambda - 1$ are units in $\mathcal{O}(S')$.

Lemma 3.7. *The morphism f is smooth and proper of relative dimension 1, with geometrically connected fibers and a 0-section. Its total space X is a smooth quasi-compact separated dagger surface. The natural model over $\mathbf{Z}_p\langle T\rangle$ has special fiber $\overline{E} \times \mathbf{A}^1_{\mathbf{F}_p}$.*

Proof. The discriminant of the Legendre equation is $16\lambda^2(1-\lambda)^2$, a unit on the indicated neighborhood. The point $[0:1:0]$ gives the section. Our lemma follows. □

Theorem 3.8 (Counterexample)**.** *X contradicts Conjecture 1.1 in degree 1.*

Proof. It suffices to show that the map (3.3) is not surjective for X and $r = i = 1$. By Proposition 3.12 below, $H^1_{\mathrm{HK}}(X_C)$ is a rank-2 isocrystal over $\breve{C}$ with slopes $0, 1$ and $N = 0$, and $F^1\mathrm{DR}^1(X) = t\mathrm{DR}^1(X)$. Over $\breve{C}$ choose a basis e_0, e_1 of $H^1_{\mathrm{HK}}(X_C)$ with $\varphi(e_0) = e_0, \varphi(e_1) = pe_1$. The Hyodo–Kato isomorphism identifies $\mathrm{DR}^1(X)/t\mathrm{DR}^1(X)$ with $Ce_0 \oplus Ce_1$. It follows from the proof of Proposition 3.12 that, in these coordinates, $\pi^{1,1}$ becomes

$$\pi^{1,1} : (\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p}e_0 \oplus (\mathbf{B}^+_{\mathrm{cr}})^{\varphi=1}e_1 \longrightarrow Ce_0 \oplus Ce_1, \quad (b, a) \mapsto (\theta(b), a)$$

and is clearly not surjective (since $(\mathbf{B}^+_{\mathrm{cr}})^{\varphi=1} = \mathbf{Q}_p$: the cokernel is actually $(C/\mathbf{Q}_p)e_1$). □

Remark 3.9. (i) It follows from the above computations that $\mathrm{Coker}\,\pi^{1,1}$ is an almost-C representation of Dimension $(1, -1)$ (it is not $C/\mathbf{Q}_p$ as one could infer from the above, but rather $C/\mathbf{Q}_p(\eta)$ for an unramified character η, because the basis e_0, e_1 only exists over $\breve{C}$, not $\mathbf{Q}_p$).

(ii) If we replace the above family by a small enough disc of curves with potentially multiplicative reduction (for example, choose $a \in p\mathbf{Z}_p \cup (1 + p\mathbf{Z}_p) \cup \frac{1}{p}\mathbf{Z}_p^\star$, $a \neq 0, 1$), the results are exactly the same.

(iii) If, instead, we take a small enough disc of curves with supersingular reduction, then we have $\mathrm{Dim}(\mathrm{Coker}\,\pi^{1,1}) = (1,-2)$, the difference being that $\pi^{1,1}$ is injective in that case, whereas its kernel is $t\mathbf{Q}_p e_0$ in the other cases.

Remark 3.10 (Negative heights)**.** Let $X \to S$ be the Legendre family of Section 3 (or its supersingular variant, Remark 3.9(iii)) over a sufficiently small dagger disc, and let $Y = X^{\times_S m}$, $m \geq 1$. Then we have

$$\mathrm{ht}\, H^{2m}_{\text{proét}}(Y_C, \mathbf{Q}_p(2m)) = \begin{cases} 1-m & \text{in the ordinary case,} \\ 1-2m & \text{in the supersingular case.} \end{cases}$$

Hence, in the ordinary case, $X \times_S X$ already gives an example of height -1 in degree 4. In the supersingular case, X itself gives an example of height -1 in degree 2.

Indeed, write $\mathcal{H}^j$ for the relative de Rham cohomology of X/S. By relative Künneth, the relative de Rham cohomology of Y/S in degree $2m$ is $(\mathcal{H}^2)^{\otimes m}$, and in degree $2m-1$ it is the direct sum of m copies of $\mathcal{H}^1 \otimes (\mathcal{H}^2)^{\otimes(m-1)}$. The class ℓ of the zero section generates $\mathcal{H}^2$ and is a horizontal Tate class of Hodge degree 1: we have $F^1\mathcal{H}^2 = \mathcal{H}^2, F^2\mathcal{H}^2 = 0$ and $\varphi(\ell) = p\ell$ in Hyodo–Kato cohomology. Thus, by Lemma 4.7 below, each factor $\mathcal{H}^2$ contributes a Tate twist to the absolute Hodge filtration and Hyodo–Kato cohomology. Hence, on the geometric Fargues–Fontaine curve we have

$$\mathcal{E}^{2m}(Y) \simeq \mathcal{O}_{X_{\mathrm{FF}}}, \qquad \mathcal{E}^{2m-1}(Y) \simeq \mathcal{E}^1(X)^{\oplus m}.$$

By the height formula in the proof of [6, Prop. 7.10], the desired height is the sum of the heights of $H^0(X_{\mathrm{FF}}, \mathcal{E}^{2m}(Y))$ and $H^1(X_{\mathrm{FF}}, \mathcal{E}^{2m-1}(Y))$. The first has height 1. By Remark 3.4, the second is m times the height of $\mathrm{Coker}\,\pi^{1,1}(X)$, which is -1 in the ordinary case and -2 in the supersingular case by Remark 3.9.

Remark 3.11. Consider the whole algebraic Legendre family

$$f\colon \mathcal{E} \longrightarrow U = \mathbb{P}^1_{\mathbf{Q}_p} \smallsetminus \{0,1,\infty\}, \qquad y^2 z = x(x-z)(x-\lambda z).$$

The total space $\mathcal{E}_C^{\mathrm{an}}$ satisfies Conjecture 1.1 by [6, Th. 8.1], since it is the analytification of a smooth algebraic variety. Thus the conjecture holds for the whole algebraic family and fails for its restrictions to sufficiently small dagger or open discs by Corollary 4.14 (or Theorem 3.8 and Remark 3.9).

It follows that the validity of Conjecture 1.1 for a variety X does not imply its validity for open analytic subspaces of X.

Proposition 3.12. *Set* $M = H^1_{\mathrm{HK}}(X_C)$ *and* $\mathrm{DR}^1 = \mathrm{DR}^1(X)$*. We have*

$$H^1_{\mathrm{dR}}(X) \xrightarrow{\sim} H^1_{\mathrm{dR}}(E), \quad F^1 H^1_{\mathrm{dR}}(X) = 0; \tag{3.13}$$

$$\mathrm{DR}^1 \simeq (\mathbf{B}^+_{\mathrm{dR}})^2, \quad F^1\mathrm{DR}^1 = t\mathrm{DR}^1; \tag{3.14}$$

$$H^1_{\mathrm{HK}}(X_C) \xrightarrow{\sim} H^1_{\mathrm{HK}}(E_C), \quad N = 0, \quad \mathrm{Slopes}_\varphi(M) = \{0,1\}. \tag{3.15}$$

Proof. The first part of (3.13) is proved in Proposition 3.20, the second part in Lemma 3.24; the first part of (3.14) is a consequence of Proposition 3.20, the second part is Corollary 3.26; finally (3.15) is proved in Proposition 3.28. □

3.2. **Trivialization of the Gauss–Manin connection.** Let $\mathcal{H} := \mathrm{R}^1 f_* \Omega^\bullet_{X/S}$ and let $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_S} \Omega^1_{S/\mathbf{Q}_p}$ be the Gauss–Manin connection. We will show that $(\mathcal{H}, \nabla)$ is analytically trivial.

Let $\omega = \mathrm{d}x/(2y), \eta = x\,\mathrm{d}x/(2y)$; the latter represents a differential of the second kind. The classes (ω, η) form a basis of $\mathcal{H}$, and the relative Hodge line is $\mathcal{O}_S\omega$.

Lemma 3.16. *We have*

$$\nabla\omega = \Big(-\frac{1}{2(\lambda-1)}\,\omega + \frac{1}{2\lambda(\lambda-1)}\,\eta\Big) \otimes d\lambda, \quad \nabla\eta = \Big(-\frac{1}{2(\lambda-1)}\,\omega + \frac{1}{2(\lambda-1)}\,\eta\Big) \otimes d\lambda. \tag{3.17}$$

Hence, the connection matrix is

$$A(\lambda) = \frac{1}{2(\lambda-1)}\begin{pmatrix} -1 & -1 \\ \lambda^{-1} & 1 \end{pmatrix}.$$

Proof. Work over $\mathbf{Q}_p(\lambda)$ and set $P(x) := x(x-1)(x-\lambda)$. We have

$$\partial_\lambda\left(\frac{x^k \mathrm{d}x}{2y}\right) = \frac{x^k x(x-1)}{4y^3}\,\mathrm{d}x, \qquad k = 0, 1.$$

For any rational function $R(x)$, we have

$$\mathrm{d}_x(R/y) = \frac{R'P - \frac{1}{2}RP'}{y^3}\,\mathrm{d}x.$$

Using the above, we compute:

$$\partial_\lambda\omega = -\frac{1}{2(\lambda-1)}\omega + \frac{1}{2\lambda(\lambda-1)}\eta + \mathrm{d}_x\left(\frac{x-x^2}{2\lambda(\lambda-1)y}\right),$$
$$\partial_\lambda\eta = -\frac{1}{2(\lambda-1)}\omega + \frac{1}{2(\lambda-1)}\eta + \mathrm{d}_x\left(\frac{x-x^2}{2(\lambda-1)y}\right).$$

Passing to de Rham cohomology classes yields (3.17), as wanted. □

By (3.17), for a general section $s = (\omega, \eta)w$, with $w = {}^t(u, v)$ we have:

$$\nabla(u\omega + v\eta) = \omega \otimes du + \eta \otimes dv + u\nabla\omega + v\nabla\eta.$$

The section s is horizontal precisely when $dw + A(\lambda)w\,d\lambda = 0$, that is, $dw/d\lambda = -A(\lambda)w$. It follows that, if $G = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ where $\alpha\omega+\gamma\eta, \beta\omega+\delta\eta$ form a basis of horizontal sections, and if $U = \lambda - a = p^2T$, then G satisfies $G'(U) = -A(a+U)G(U)$.

Write $A(a+U) = \sum_{n\geq 0} A_n U^n$. All the A_n belong to $M_2(\mathbf{Z}_p)$, because 2, $a+U$, and $a-1+U$ are units in $\mathbf{Z}_p[[U]]$.

Lemma 3.18. *Let $G = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in M_2(\mathbf{Q}_p[[U]])$ be the unique formal matrix solution of*

$$G'(U) = -A(a+U)G(U), \qquad G(0) = I_2.$$

Then G converges for $v_p(U) > \frac{1}{p-1}$, has determinant 1, and $\alpha\omega+\gamma\eta, \beta\omega+\delta\eta$ form a horizontal basis of $\mathcal{H}$ over a strict neighborhood of S.

Proof. Write $G(U) = \sum_{n\geq 0} G_n U^n$. The above differential equation gives the relation

$$(n+1)G_{n+1} = -\sum_{j=0}^{n} A_j G_{n-j}$$

from which we deduce, by induction, that $n!G_n \in M_2(\mathbf{Z}_p)$: indeed,

$$(n+1)!G_{n+1} = -\sum_{j=0}^{n} A_j \frac{n!}{(n-j)!}\big((n-j)!G_{n-j}\big).$$

As $v_p(n!)/n \to \frac{1}{p-1}$, the series converges on the asserted disc. The trace of A is 0, so differentiation of the determinant gives $(\det G)' = 0$ and $\det G = 1$.

The series converge for $v_p(T) > \frac{1}{p-1} - 2$. In particular, G and G^{-1} are analytic on a disc strictly larger than the closed unit disc. Multiplication by $G(p^2T)$ identifies $\mathcal{H}$, with its connection, with the constant bundle with fiber $\mathcal{H}_0$. □

The de Rham Leray spectral sequence for the smooth proper morphism f is

$$E_2^{a,b} = H^a_{\mathrm{dR}}\big(S, R^b f_* \Omega^\bullet_{X/S}\big) \Longrightarrow H^{a+b}_{\mathrm{dR}}(X). \tag{3.19}$$

Here cohomology on the base is computed using the Gauss–Manin connection.

Proposition 3.20. *Restriction to $T = 0$ induces isomorphisms*

$$j^* : H^i_{\mathrm{dR}}(X) \xrightarrow{\sim} H^i_{\mathrm{dR}}(E) \quad \textit{for every } i.$$

In particular, the dimensions in degrees $0, 1, 2$ are $1, 2, 1$, and the other groups vanish. The same claim holds after base change from $\mathbf{Q}_p$ to C.

Proof. The relative cohomology bundles are 0 outside degrees $0, 1, 2$. In degree 0 the bundle is just $(\mathcal{O}, \mathrm{d})$; in degree 1 it is constant as a connection by Lemma 3.18; and in degree 2 it is a trivial rank-1 connection, generated by the relative class of the 0-section. Since $H^i_{\mathrm{dR}}(S) = 0$, for $i > 0$, each of these three connections has $H^1_{\mathrm{dR}} = 0$ on the base. Thus (3.19) is supported in the column $a = 0$. Evaluation of a horizontal section at $T = 0$ identifies that column with the cohomology of E, proving the proposition. □

Remark 3.21. In degree 1, the isomorphisms in Proposition 3.20 are not strict for the Hodge filtration; we will show this in the next computation.

3.3. **The Hodge filtration.** We will show that $F^1 H^1_{\mathrm{dR}}(X) = 0$. We have an exact sequence

$$0 \longrightarrow f^* \Omega^1_{S/\mathbf{Q}_p} \longrightarrow \Omega^1_{X/\mathbf{Q}_p} \longrightarrow \Omega^1_{X/S} \longrightarrow 0.$$

Writing $\omega_f = f_* \Omega^1_{X/S}$ and using $f_* \mathcal{O}_X = \mathcal{O}_S$, its pushforward starts as

$$0 \longrightarrow \Omega^1_{S/\mathbf{Q}_p} \longrightarrow f_* \Omega^1_{X/\mathbf{Q}_p} \longrightarrow \omega_f \xrightarrow{\mathrm{KS}} (R^1 f_* \mathcal{O}_X) \otimes \Omega^1_{S/\mathbf{Q}_p}, \tag{3.22}$$

where KS denotes the Kodaira–Spencer map.

Lemma 3.23. *The map* KS *is an isomorphism of line bundles. Hence, $f_* \Omega^1_{X/\mathbf{Q}_p} \simeq \Omega^1_{S/\mathbf{Q}_p}$.*

Proof. The Kodaira–Spencer map is the composite (we identify $\mathcal{H}/\omega_f \simeq \mathrm{R}^1 f_* \mathcal{O}_X$)

$$\omega_f \to \mathcal{H} \xrightarrow{\nabla} \mathcal{H} \otimes \Omega^1_{S/\mathbf{Q}_p} \xrightarrow{\mathrm{pr} \otimes 1} (\mathcal{H}/\omega_f) \otimes \Omega^1_{S/\mathbf{Q}_p}.$$

The first formula of (3.17) gives

$$\nabla \omega \bmod \omega_f = \frac{1}{2\lambda(\lambda - 1)} \, \overline{\eta} \otimes \mathrm{d}\lambda = \frac{p^2}{2\lambda(\lambda - 1)} \, \overline{\eta} \otimes \mathrm{d}T.$$

Our lemma follows since $p^2/(2\lambda(\lambda - 1))$ is a unit in $\mathcal{O}(S)$. □

Lemma 3.24. *We have $F^1 H^1_{\mathrm{dR}}(X) = 0$.*

Proof. We have $F^1 H^1_{\mathrm{dR}}(X) = \mathrm{Im}(\Omega^1(X)^{d=0} \to H^1_{\mathrm{dR}}(X))$. By Lemma 3.23, every global 1-form is pulled back from S. Such a form is closed since S is a curve, and is exact since $H^1_{\mathrm{dR}}(S) = 0$; its de Rham class on X is therefore 0. □

Remark 3.25. There is no contradiction between Lemma 3.24 and the 1-dimensional F^1 on each fiber: a relative invariant differential does not extend to a global 1-form on X when the Kodaira–Spencer map is nonzero.

Corollary 3.26. $F^1 \mathrm{DR}^1 = t\mathrm{DR}^1$.

Proof. This is a direct consequence of (3.2), granted Lemma 3.24. □

3.4. **Hyodo–Kato cohomology.** We have the functorial and compatible Hyodo–Kato isomorphisms (see [6, Prop. 4.24, Prop. 4.27, Cor. 5.12])

$$\begin{aligned} H^i_{\mathrm{HK}}(X_C) \otimes^{\square}_{\breve{C}} C &\xrightarrow{\sim} H^i_{\mathrm{dR}}(X_C), \\ H^i_{\mathrm{HK}}(X_C) \otimes^{\square}_{\breve{C}} \mathbf{B}^+_{\mathrm{dR}} &\xrightarrow{\sim} H^i_{\mathrm{dR}}(X) \otimes^{\square}_{K} \mathbf{B}^+_{\mathrm{dR}}. \end{aligned} \tag{3.27}$$

Proposition 3.28. *$j : E_C \hookrightarrow X_C$ induces an isomorphism of (φ, N)-modules:*

$$j^* : H^i_{\mathrm{HK}}(X_C) \xrightarrow{\sim} H^i_{\mathrm{HK}}(E_C) \quad \textit{for every } i.$$

Proof. By functoriality of (3.27) this follows from Proposition 3.20. □

Combined with the hypothesis that E has good ordinary reduction, this yields (3.15) in Proposition 3.12.

Remark 3.29 (Restriction to the open disc)**.** Let $S^\circ = \{v_p(T) > 0\}$ and let $X^\circ \to S^\circ$ be the restriction of (3.6). This is a smooth proper elliptic fibration over a Stein curve, and the total space is partially proper. Since the Poincaré lemma holds for the open disc, the same arguments as in the preceding proofs yield the following result.

Corollary 3.30. *X° contradicts Conjecture 1.1 in degree* 1*.*

3.5. **Alternative proof of Corollary 3.30.** We now compute $H^1_{\text{proét}}(X^\circ_C, \mathbf{Q}_p(1))$ directly, using Kummer theory and Serre–Tate theory, and derive from that another proof of Corollary 3.30.

Proposition 3.31. *Let $\mathcal{V} = \mathrm{R}^1 f_* \mathbf{Q}_p(1)$, a $\mathbf{Q}_p$-local system on S°_C.*
(i) *We have $H^0(S^\circ_C, \mathcal{V}) \simeq \mathbf{Q}_p(1)$.*
(ii) *There is an exact sequence:*

$$0 \longrightarrow \mathcal{O}(S^\circ_C)/C \longrightarrow H^1_{\text{proét}}(X^\circ_C, \mathbf{Q}_p(1)) \longrightarrow \mathbf{Q}_p(1) \longrightarrow 0. \tag{3.32}$$

Proof. Step 1: The implication (i)⇒(ii). Since f is proper and smooth, the sheaves $\mathrm{R}^b f_* \mathbf{Q}_p(1)$ are $\mathbf{Q}_p$-local systems on S°_C [11, Th. 10.5.1]. We have $\mathrm{R}^0 f_* \mathbf{Q}_p(1) = \mathbf{Q}_p(1)$, because the fibers of f are geometrically connected, and $\mathcal{V} := \mathrm{R}^1 f_* \mathbf{Q}_p(1)$ is the rational Tate module $V_p(X^\circ_C/S^\circ_C)$ of the family. Consider the Leray spectral sequence

$$E_2^{a,b} = H^a_{\text{proét}}(S^\circ_C, \mathrm{R}^b f_* \mathbf{Q}_p(1)) \Longrightarrow H^{a+b}_{\text{proét}}(X^\circ_C, \mathbf{Q}_p(1)).$$

Since $H^2_{\text{proét}}(S^\circ_C, \mathbf{Q}_p(1)) = 0$ (because S° is a Stein curve, see [2, Th. 1.8]), we get an exact sequence

$$0 \longrightarrow H^1_{\text{proét}}(S^\circ_C, \mathbf{Q}_p(1)) \xrightarrow{f^*} H^1_{\text{proét}}(X^\circ_C, \mathbf{Q}_p(1)) \longrightarrow H^0(S^\circ_C, \mathcal{V}) \longrightarrow 0. \tag{3.33}$$

Since $H^i_{\mathrm{dR}}(S^\circ_C) = 0$ for $i > 0$, we have $H^1_{\text{proét}}(S^\circ_C, \mathbf{Q}_p(1)) \simeq \mathcal{O}(S^\circ_C)/C$ by [2, Th. 1.8]. Thus (ii) follows from (i) and (3.33). Hence we just have to prove (i).

Step 2: An integral model. Put $W = W(\overline{\mathbf{F}}_p)$ and $R = W[[T]]$, with its (p,T)-adic topology. The Legendre equation with $\lambda = a + p^2 T$ defines an elliptic curve $\mathfrak{X}/R$ with ordinary closed fiber $\overline{E}$. Fix identifications of the multiplicative and étale parts of $\overline{E}[p^\infty]$ with $\boldsymbol{\mu}_{p^\infty}$ and $\mathbf{Q}_p/\mathbf{Z}_p$. Then $G = \mathfrak{X}[p^\infty]$ is the extension

$$0 \longrightarrow \boldsymbol{\mu}_{p^\infty} \longrightarrow G \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow 0 \tag{3.34}$$

with Serre–Tate parameter $q \in 1 + (p,T)R$ [9, Th. 2.1]. In particular, q is an analytic unit on S°_C.

Step 3: The extension class. The rational Tate module of (3.34) gives

$$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow \mathcal{V} \longrightarrow \mathbf{Q}_p \longrightarrow 0$$

on S°_C. Its boundary sends 1 to the Kummer class $\kappa(q)$, since the fiber of $G[p^n] \to \mathbf{Z}/p^n$ over 1 is the torsor of p^n-th roots of q. The natural map $H^1_{\text{proét}}(S^\circ_C, \mathbf{Q}_p(1)) \to \Omega^1(S^\circ_C)$ sends $\kappa(q)$ to $\mathrm{d}q/q$ [2]. Hence, if q is nonconstant, the boundary is injective and $H^0(S^\circ_C, \mathcal{V}) = \mathbf{Q}_p(1)$.

Step 4: Nonconstancy of q. Since the Legendre j-invariant $J(\lambda)$ is a nonconstant rational function, choose $t \in pW$ such that $J(a + p^2 t) \neq J(a)$. The fibers $\mathfrak{X}_t$ and $\mathfrak{X}_0$ are lifts of the same $\overline{E}$ to W, with Serre–Tate parameters $q(t)$ and $q(0)$, respectively. If $q(t) = q(0)$, these lifts would be isomorphic by [9, Th. 2.1], contradicting that their j-invariants are not equal. Thus q is nonconstant on S°_C, proving (i). □

Remark 3.35. (1) By (3.32), the height of[3] $H^1_{\text{proét}}(X^\circ_C, \mathbf{Q}_p(1))$ is 1, whereas, if Conjecture 1.1 were true, it should be $\dim_C H^1_{\mathrm{dR}}(X^\circ_C) = 2$ according to [6, Prop. 7.10]. This gives a second proof of Corollary 3.30.

(2) The two computations agree. The basic comparison sequence [6, Cor. 7.9], [5] in degree 1, for $r = 1$, reads

$$\mathrm{HK}^{1,0}(X^\circ) \oplus F^{1,0}(X^\circ) \to \mathrm{DR}^0(X^\circ) \to H^1_{\text{proét}}(X^\circ_C, \mathbf{Q}_p(1)) \to \mathrm{HK}^{1,1}(X^\circ) \oplus F^{1,1}(X^\circ) \to \mathrm{DR}^1(X^\circ).$$

The first map is surjective, since $\mathrm{DR}^0(X^\circ) = \mathbf{B}^+_{\mathrm{dR}}$, $F^{1,0}(X^\circ) = t\mathbf{B}^+_{\mathrm{dR}}$, and $\theta\big((\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p}\big) = C$. The kernel of $F^{1,1}(X^\circ) \to \mathrm{DR}^1(X^\circ)$ is the cokernel of $H^0(\mathrm{DR}(X^\circ)) \to H^0(\mathrm{DR}(X^\circ)/F^1) = \mathcal{O}(X^\circ_C)$, i.e., it is $\mathcal{O}(S^\circ_C)/C$; its image is $F^1\mathrm{DR}^1 = t\,\mathrm{DR}^1$ by Remark 3.29. We get the exact sequence

$$0 \longrightarrow \mathcal{O}(S^\circ_C)/C \longrightarrow H^1_{\text{proét}}(X^\circ_C, \mathbf{Q}_p(1)) \longrightarrow \operatorname{Ker}\pi^{1,1} \longrightarrow 0,$$

and $\operatorname{Ker}\pi^{1,1} = \mathbf{Q}_p\, t\, e_0 \simeq \mathbf{Q}_p(1)$ by the proof of Theorem 3.8. Thus the two computations give the same extension. On the étale side, the quotient $\mathbf{Q}_p(1) = H^0(S^\circ_C, \mathcal{V})$ is the rational Tate module of the multiplicative part $\boldsymbol{\mu}_{p^\infty}$ of G. On the de Rham side, it is $\operatorname{Ker}\pi^{1,1}$.

4. A local criterion

We now give a necessary and sufficient condition for Conjecture 1.1 to hold for a family over a small disc.

Let U be a smooth rigid analytic curve over K, and let $f : \mathcal{X} \to U$ be smooth and proper. By Scholze's relative comparison theorem [10, Th. 8.8], whose hypothesis on the higher direct images of the constant $\widehat{\mathbf{Z}}_p$-local system follows from [11, Th. 10.5.1], the relative Hodge–de Rham spectral sequence

$$E_1^{a,b} = R^b f_* \Omega^a_{\mathcal{X}/U} \Longrightarrow \mathcal{H}^{a+b} := R^{a+b} f_* \Omega^\bullet_{\mathcal{X}/U}$$

degenerates at E_1, and its terms are locally free. They commute with base change by proper coherent base change. Write $F^\bullet\mathcal{H}^i$ for the relative Hodge filtration and ∇ for the Gauss–Manin connection. Griffiths transversality induces the Higgs maps

$$\theta^a_i : \quad \mathrm{gr}^a_F \mathcal{H}^i \longrightarrow \mathrm{gr}^{a-1}_F \mathcal{H}^i \otimes_{\mathcal{O}_U} \Omega^1_{U/K}.$$

Their vanishing is equivalent to

$$\nabla(F^a\mathcal{H}^i) \subseteq F^a\mathcal{H}^i \otimes_{\mathcal{O}_U} \Omega^1_{U/K}, \qquad \text{for all } a, i. \tag{4.1}$$

4.1. **The Hodge filtration.** Let S be a dagger affinoid in U or a Stein open subspace of U, and set $X = \mathcal{X} \times_U S$.

Lemma 4.2. *For every $a, i \geq 0$, the edge map induces a surjection*

$$H^i(X, \Omega^{\geq a}_{X/K}) \longrightarrow \Gamma(S, F^a\mathcal{H}^i)^{\nabla=0}. \tag{4.3}$$

Proof. Since the base is a curve, we have an exact sequence

$$0 \longrightarrow f^{-1}\Omega^1_{S/K} \otimes_{f^{-1}\mathcal{O}_S} \Omega^{\geq a-1}_{X/S}[-1] \longrightarrow \Omega^{\geq a}_{X/K} \longrightarrow \Omega^{\geq a}_{X/S} \longrightarrow 0. \tag{4.4}$$

Relative Hodge–de Rham degeneration identifies $\mathrm{R}^i f_* \Omega^{\geq a}_{X/S} \xrightarrow{\sim} F^a\mathcal{H}^i \subseteq \mathcal{H}^i$. Hence, coherent acyclicity on S identifies the relevant part of the cohomology sequence of (4.4) with

$$H^i(X, \Omega^{\geq a}_{X/K}) \longrightarrow H^0(S, F^a\mathcal{H}^i) \xrightarrow{\nabla} H^0(S, F^{a-1}\mathcal{H}^i \otimes_{\mathcal{O}_S} \Omega^1_{S/K}).$$

[3] To justify this statement, one has to upgrade (3.32) to the category of qBC's or define the height for their almost-C-representation avatars (if K is a finite extension of $\mathbf{Q}_p$). Recall that, for an almost C-representation W, we have $\sum_{i=0}^2 (-1)^i \dim_{\mathbf{Q}_p} H^i(G_K, W) = -[K : \mathbf{Q}_p]\mathrm{ht}(W)$, see [8] (this follows from Tate's Galois cohomology computations). In our case, this would make no sense since H^0 and H^1 are infinite dimensional. But $\cup \log \chi_{\mathrm{cycl}}$ induces a map $\ell : H^0(G_K, W) \to H^1(G_K, W)$, with finite dimensional kernel and cokernel. Hence we can define $\mathrm{ht}(W)$ by the formula $\dim_{\mathbf{Q}_p} \operatorname{Coker}\ell - \dim_{\mathbf{Q}_p} \operatorname{Ker}\ell - \dim_{\mathbf{Q}_p} H^2(G_K, W) = [K : \mathbf{Q}_p]\mathrm{ht}(W)$.

The map ∇ is the filtered Gauss–Manin connection. Its target injects into $H^0(S, \mathcal{H}^i \otimes_{\mathcal{O}_S} \Omega^1_{S/K})$. Our lemma follows. □

Hypothesis 4.5. We fix $s_0 \in U(K)$, and $S \subset U$ is a dagger disc or an open disc centered at s_0, small enough for each of the finitely many connections $(\mathcal{H}^i, \nabla)$ to admit a horizontal trivialization on a neighborhood of S:

$$\mathcal{H}^i|_S \simeq \mathcal{O}_S \otimes H^0(S, \mathcal{H}^i)^{\nabla=0}. \tag{4.6}$$

We set $X := \mathcal{X} \times_U S$ (if S is a dagger disc, X is a dagger variety, whose dagger structure comes from the $\mathcal{X} \times_U S'$, for S' running through the discs centered at s_0 of radius strictly bigger than that of S).

Such discs exist: the argument is the same as for Lemma 3.18, i.e., choose S_0 a small enough disc around s_0 such that $\mathcal{H}^i$ admits a basis over $\mathcal{O}(S_0)$; write the equation $\nabla = 0$ in this basis; one gets a differential equation $G' = -AG$, and the formal solution with $G(s_0) = 1$ converges and is invertible on some disc around s_0. We will use the following properties of such an S: it is geometrically connected, $S(K)$ is Zariski dense in S, coherent sheaves on S are acyclic, and $H^i_{\mathrm{dR}}(S) = 0$ for $i > 0$ (Poincaré lemma).

If $s \in S(K)$, denote by X_s the fiber above s, and by j_s the closed embedding $X_s \hookrightarrow X$.

Lemma 4.7. *Let S and X be as in Hypothesis 4.5.*

(i) $F^a H^i_{\mathrm{dR}}(X) \simeq H^0(S, F^a\mathcal{H}^i)^{\nabla=0}$ *for all a.*

(ii) j_s^* *induces isomorphisms* $j_s^* : H^i_{\mathrm{dR}}(X) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X_s)$ *and* $j_s^* : H^i_{\mathrm{HK}}(X_C) \xrightarrow{\sim} H^i_{\mathrm{HK}}(X_{s,C})$.

Proof. Set $V^j := H^0(S, \mathcal{H}^j)^{\nabla=0}$. Consider the Leray spectral sequence $E_2^{i,j} = H^i_{\mathrm{dR}}(S, \mathcal{H}^j) \Rightarrow H^{i+j}_{\mathrm{dR}}(X)$. By (4.6), $H^i_{\mathrm{dR}}(S, \mathcal{H}^j) = H^i_{\mathrm{dR}}(S) \otimes_K V^j$. This vanishes for $i > 0$ by the Poincaré lemma for the dagger and open discs. Hence the edge map

$$e : \ H^i_{\mathrm{dR}}(X) \longrightarrow H^0_{\mathrm{dR}}(S, \mathcal{H}^i) = H^0(S, \mathcal{H}^i)^{\nabla=0}$$

is an isomorphism. Its composite with evaluation at s is j_s^*, which proves (i) for $a = 0$.

The map e is induced by $\Omega^\bullet_{X/K} \to \Omega^\bullet_{X/S}$, together with the identification $H^i(X, \Omega^\bullet_{X/S}) = H^0(S, \mathcal{H}^i)$ given by coherent acyclicity on S. Consequently, the restriction of e to the image of $H^i(X, \Omega^{\geq a}_{X/K})$ is induced by $\Omega^{\geq a}_{X/K} \to \Omega^{\geq a}_{X/S}$. That is, it is the edge map (4.3) followed by the inclusion $H^0(S, F^a\mathcal{H}^i) \subseteq H^0(S, \mathcal{H}^i)$. By Lemma 4.2, its image is $H^0(S, F^a\mathcal{H}^i)^{\nabla=0}$. This proves (i).

For H^i_{dR}, (ii) follows from (i), (4.6) and the fact that the fiber at s of $\mathcal{H}^i$ is $H^i_{\mathrm{dR}}(X_s)$, by construction. The statement for H^i_{HK} follows from that for H^i_{dR} and functoriality of the Hyodo–Kato isomorphism. □

4.2. **Acyclicity of vector bundles and vanishing of the Higgs map.** Let

$$\mathcal{E}^i(X) := \mathcal{E}(H^i_{\mathrm{HK}}(X_C), F^0(H^i_{\mathrm{dR}}(X) \otimes_K \mathbf{B}_{\mathrm{dR}}))$$

be the[4] vector bundle on the Fargues–Fontaine curve X_{FF} attached to $(H^i_{\mathrm{HK}}(X_C), H^i_{\mathrm{dR}}(X))$. Conjecture 1.1 for r, i is equivalent to $H^1(X_{\mathrm{FF}}, \mathcal{E}^j(X)) = 0$, for $j = i-1, i$; i.e., $\mathcal{E}^j(X)$ *is acyclic* for $j = i, i-1$.

Now, by Lemma 4.7, j_s^* induces isomorphisms $j_s^* : H^i_{\mathrm{dR}}(X) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X_s)$ and $j_s^* : H^i_{\mathrm{HK}}(X_C) \xrightarrow{\sim} H^i_{\mathrm{HK}}(X_{s,C})$, and sends $F^a H^i_{\mathrm{dR}}(X)$ to $F^a H^i_{\mathrm{dR}}(X_s)$ (but the induced map has no reason to be an isomorphism, a priori). This induces an exact sequence of coherent sheaves on X_{FF}:

$$0 \to \mathcal{E}^i(X) \to \mathcal{E}^i(X_s) \to Z^i \to 0$$

where Z^i is supported on the point x_∞ (defined by the equation $t = 0$), with

$$H^0(X_{\mathrm{FF}}, Z^i) = F^0(H^i_{\mathrm{dR}}(X_s) \otimes_K \mathbf{B}_{\mathrm{dR}})/F^0(H^i_{\mathrm{dR}}(X) \otimes_K \mathbf{B}_{\mathrm{dR}})$$

[4]The vector bundle $\mathcal{E}^i(X)$ is isomorphic to the vector bundle $\mathcal{E}^{r,i}(X)$ defined in the introduction; they only differ by a Tate twist if one takes Galois action into account.

(a $\mathbf{B}^+_{\mathrm{dR}}$-module killed by a power of t).

But X_s is proper, hence $\mathcal{E}^i(X_s)$ is pure, of slope 0, so that $H^0(X_{\mathrm{FF}}, \mathcal{E}^i(X_s))$ is finite dimensional over $\mathbf{Q}_p$, and $H^1(X_{\mathrm{FF}}, \mathcal{E}^i(X_s)) = 0$. It follows that $\mathcal{E}^i(X)$ is acyclic if and only if $H^0(X_{\mathrm{FF}}, Z^i) = 0$ or, equivalently, if and only if $j_s^\star : F^a H^i_{\mathrm{dR}}(X) \to F^a H^i_{\mathrm{dR}}(X_s)$ is an isomorphism, for all a. Hence we have proved the following result:

Lemma 4.8. *The following conditions are equivalent:*

(i) $\mathcal{E}^i(X)$ *is acyclic.*

(ii) *There exists* $s \in S(K)$*, such that* $j_s^\star : F^a H^i_{\mathrm{dR}}(X) \to F^a H^i_{\mathrm{dR}}(X_s)$ *is an isomorphism, for all* a*.*

(iii) *For all* $s \in S(K)$*,* $j_s^\star : F^a H^i_{\mathrm{dR}}(X) \to F^a H^i_{\mathrm{dR}}(X_s)$ *is an isomorphism, for all* a*.*

It follows that condition (iii) of Lemma 4.8 for a is also equivalent to:

(iv) The natural injection $\mathcal{O}_S \otimes H^0(S, F^a\mathcal{H}^i)^{\nabla=0} \to F^a\mathcal{H}^i$ is an isomorphism.

(That it is injective is a general fact in the theory of differential equations; that it is surjective if and only if (iii) is true follows from the fact that the fiber of $F^a\mathcal{H}^i$ at s is $F^a H^i_{\mathrm{dR}}(X_s)$.)

Now, since $\mathcal{O}_S \otimes H^0(S, \mathcal{H}^i)^{\nabla=0} \to \mathcal{H}^i$ is an isomorphism, (iv) is equivalent to

(v) $F^a\mathcal{H}^i$ is stable by ∇.

This in turn is equivalent to the vanishing of the Higgs map θ_i^a. Putting everything together, we obtain the equivalence between (i) and (ii) in the following theorem:

Theorem 4.9. *Let* S *and* X *be as in Hypothesis 4.5 and let* $i \geq 1$*. The following conditions are equivalent:*

(i) *Conjecture 1.1 is true for* X *in degree* i *(for one or, equivalently, for all* $r \geq i$*).*

(ii) *All Higgs maps* θ_j^a *are* 0 *on* S*, for* $j = i-1, i$*.*

(iii) *The* $\mathbf{Q}_p$*-local systems* $\mathrm{R}^j f_* \mathbf{Q}_p$ *are constant on* S_C[5]*, for* $j = i-1, i$*.*

Proof. The equivalence between (i) and (ii) has been proved above; we are going to prove (i)⇒(iii) and (iii)⇒(ii). Fix $j \in \{i-1, i\}$ and $s \in S(K)$, and let $\mathcal{V}^j := \mathrm{R}^j f_* \mathbf{Q}_p$; this is a $\mathbf{Q}_p$-local system on S_C [11, Th. 10.5.1], and its fiber at s is $H^j_{\text{ét}}(X_{s,C}, \mathbf{Q}_p)$.

(i)⇒(iii). Let $r \geq i$. Since X_s is proper, $\pi^{r,j}(X_s)$ is surjective, the map $H^j_{\text{ét}}(X_{s,C}, \mathbf{Q}_p(r)) \to \mathrm{HK}^{r,j}(X_s)$ is injective, and its image is $\operatorname{Ker} \pi^{r,j}(X_s)$ [6, Cor. 6.15, Rem. 6.23(i)]. On the other hand, it follows from the exactness of (1.3) (in degree j) and from the definition (3.1) of $F^r\mathrm{DR}^j(X)$, that $\operatorname{Ker} \pi^{r,j}(X)$ is the image of $H^j_{\text{proét}}(X_C, \mathbf{Q}_p(r)) \to \mathrm{HK}^{r,j}(X)$.

Now, if (i) holds, $\mathcal{E}^j(X)$ is acyclic. Hence, by Lemmas 4.7 and 4.8, and by (3.2), $j_s^\star$ induces isomorphisms

$$\mathrm{HK}^{r,j}(X) \xrightarrow{\sim} \mathrm{HK}^{r,j}(X_s), \quad \mathrm{DR}^j(X) \xrightarrow{\sim} \mathrm{DR}^j(X_s), \quad F^r\mathrm{DR}^j(X) \xrightarrow{\sim} F^r\mathrm{DR}^j(X_s),$$

and hence also $\operatorname{Ker} \pi^{r,j}(X) \xrightarrow{\sim} \operatorname{Ker} \pi^{r,j}(X_s)$. By functoriality of the comparison maps, this implies that the map

$$j_s^\star : H^j_{\text{proét}}(X_C, \mathbf{Q}_p(r)) \longrightarrow H^j_{\text{ét}}(X_{s,C}, \mathbf{Q}_p(r))$$

is surjective (use (1.3)). But this map factors as

$$H^j_{\text{proét}}(X_C, \mathbf{Q}_p(r)) \longrightarrow H^0(S_C, \mathcal{V}^j(r)) \longrightarrow H^j_{\text{ét}}(X_{s,C}, \mathbf{Q}_p(r)),$$

where the first map is the edge map of the Leray spectral sequence, and the second is the evaluation at s. It follows that every element of the fiber of $\mathcal{V}^j$ at s extends to a global section of $\mathcal{V}^j$ on S_C; since S_C is connected, this means that $\mathcal{V}^j$ is constant. (If S is a dagger disc, $H^j_{\text{proét}}(X_C, \mathbf{Q}_p(r))$ is the colimit of the $H^j_{\text{proét}}(X'_C, \mathbf{Q}_p(r))$, for $X' = \mathcal{X} \times_U S'$ and S' running through the discs centered

[5] If S is a dagger disc, this means: on S'_C, for some disc $S' \subset U$ centered at s_0, of radius strictly bigger than that of S.

at s_0 of radius strictly bigger than that of S; as $H^j_{\text{ét}}(X_{s,C}, \mathbf{Q}_p(r))$ is finite dimensional, the above argument shows that $\mathcal{V}^j$ is constant on S'_C, if S' is small enough.)

(iii)⇒(ii). By [10, Th. 8.8], the local system $\mathrm{R}^j f_* \mathbf{Q}_p$ on S is de Rham and the filtered module with integrable connection associated to it, in the sense of [10, Def. 7.5], is $(\mathcal{H}^j, \nabla, F^\bullet)$. Assume that $\mathcal{V}^j$ is constant on S_C (replace S by S' in what follows, if S is a dagger disc). Since S is geometrically connected, $\mathrm{R}^j f_* \mathbf{Q}_p$ is then the pullback to S of the representation $V := H^0(S_C, \mathcal{V}^j) \simeq H^j_{\text{ét}}(X_{s,C}, \mathbf{Q}_p)$ of G_K. This representation is de Rham; hence $\mathrm{R}^j f_* \mathbf{Q}_p$ is also associated to $\mathcal{O}_S \otimes_K D_{\mathrm{dR}}(V)$, endowed with the connection $d \otimes 1$ and the filtration $\mathcal{O}_S \otimes_K F^\bullet D_{\mathrm{dR}}(V)$. By uniqueness of the associated filtered module with integrable connection [10, Th. 7.6], we have $F^a\mathcal{H}^j \simeq \mathcal{O}_S \otimes_K F^a D_{\mathrm{dR}}(V)$, for all a. Hence $F^a\mathcal{H}^j$ is generated by horizontal sections; in particular, it is stable by ∇, and $\theta^a_j = 0$ for all a. □

Remark 4.10. For the family of Section 3.1, Proposition 3.31 shows directly that condition (iii) of Theorem 4.9 is not satisfied for $i = 1$: the space $H^0(S^\circ_C, \mathrm{R}^1 f_* \mathbf{Q}_p)$ is of dimension 1 and not 2. This is the étale counterpart of Lemma 3.23.

Remark 4.11 (The period map)**.** Theorem 4.9 has a simple geometric interpretation. Let S and X be as in Theorem 4.9, and let $V^i := H^0(S, \mathcal{H}^i)^{\nabla=0}$. By Lemma 4.7, we have $H^i_{\mathrm{dR}}(X) \simeq V^i$ and, for every $u \in S(K)$, evaluation at u is an isomorphism $V^i \xrightarrow{\sim} H^i_{\mathrm{dR}}(X_u)$ (horizontal transport). We denote by $F^\bullet_u V^i$ the inverse image by this isomorphism of the Hodge filtration of $H^i_{\mathrm{dR}}(X_u)$. So all the fibers have the same de Rham cohomology V^i, but their Hodge filtrations $F^\bullet_u V^i$ move with u; the dimension of $F^a_u V^i$ does not depend on u, since $F^a\mathcal{H}^i$ is a subbundle of $\mathcal{H}^i$. The map $u \mapsto F^\bullet_u V^i$, from $S(K)$ to the variety of flags of V^i of the corresponding type, is the period map of $(\mathcal{H}^i, \nabla, F^\bullet)$.

(i) *The Hodge filtration of the total space is the part common to the Hodge filtrations of all the fibers*: for all a, we have

$$F^a H^i_{\mathrm{dR}}(X) = \bigcap_{u \in S(K)} F^a_u V^i \quad \text{in } V^i. \tag{4.12}$$

Indeed, by Lemma 4.7, $v \in V^i$ belongs to $F^a H^i_{\mathrm{dR}}(X)$ if and only if, viewed as a section of $\mathcal{H}^i$, it is a section of $F^a\mathcal{H}^i$, i.e. if and only if its image in $H^0(S, \mathcal{H}^i/F^a\mathcal{H}^i)$ is 0. As $\mathcal{H}^i/F^a\mathcal{H}^i$ is locally free, and as an analytic function on S which vanishes at all points of $S(K)$ is 0, this is the case if and only if $v(u) \in F^a H^i_{\mathrm{dR}}(X_u)$ for all $u \in S(K)$, i.e. if and only if $v \in F^a_u V^i$ for all u.

Since V^i is finite dimensional, finitely many u suffice in (4.12); see the Appendix for the $\mathbf{B}^+_{\mathrm{dR}}$-version of (4.12).

(ii) Consider the family of Section 3.1. Let ω_0, η_0 be the classes of ω, η in $H^1_{\mathrm{dR}}(E)$, and identify V^1 with $H^1_{\mathrm{dR}}(E)$ by evaluation at $T = 0$. Let $G = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ be as in Lemma 3.18, viewed as a function of λ via $U = \lambda - a$. A horizontal section $c_1(\alpha\omega + \gamma\eta) + c_2(\beta\omega + \delta\eta)$ has value $c_1\omega_0 + c_2\eta_0$ at $T = 0$, and its value at λ belongs to $F^1 = K\omega$ if and only if $c_1\gamma(\lambda) + c_2\delta(\lambda) = 0$. Hence the period map is

$$\lambda \longmapsto F^1_\lambda V^1 = K\big(\delta(\lambda)\,\omega_0 - \gamma(\lambda)\,\eta_0\big).$$

From $G' = -AG$ and $G(a) = I_2$, we get $\gamma(a) = 0$, $\delta(a) = 1$ and $\gamma'(a) = -\frac{1}{2a(a-1)} \neq 0$ (this is, up to sign, the coefficient of the Kodaira–Spencer map of Lemma 3.23). So γ is not identically 0, and $F^1_\lambda V^1 \neq K\omega_0 = F^1_a V^1$ if $\gamma(\lambda) \neq 0$. Two distinct lines of a plane meet in 0, and (4.12) gives $F^1 H^1_{\mathrm{dR}}(X) = 0$: this is Lemma 3.24.

Corollary 4.13. *Let S and X be as in Hypothesis 4.5, and let $i \geq 1$. Then Conjecture 1.1 holds for X in degree i if and only if the period maps of $\mathcal{H}^{i-1}$ and $\mathcal{H}^i$ are constant on $S(K)$. In particular, X satisfies Conjecture 1.1 if and only if all its period maps are constant.*

Proof. We use the notation of Remark 4.11. Conjecture 1.1 holds for X in degree i if and only if $\mathcal{E}^j(X)$ is acyclic for $j = i-1, i$ (see Section 4.2). Fix such a j, and $s \in S(K)$. Via the isomorphisms $H^j_{\mathrm{dR}}(X) \simeq V^j \simeq H^j_{\mathrm{dR}}(X_s)$ of Lemma 4.7, the map $j_s^\star$ becomes the identity of V^j; hence, by

Lemma 4.8, $\mathcal{E}^j(X)$ is acyclic if and only if $F^aH^j_{\mathrm{dR}}(X) = F^a_sV^j$ for all a. If this is the case, (4.12) gives $F^a_sV^j \subset F^a_uV^j$, hence $F^a_uV^j = F^a_sV^j$ since both spaces have the same dimension, for all $u \in S(K)$ and all a: the period map of $\mathcal{H}^j$ is constant. Conversely, if it is constant, (4.12) gives $F^aH^j_{\mathrm{dR}}(X) = F^a_sV^j$ for all a. □

In this language, the equivalence (i)⇔(ii) of Theorem 4.9 reflects the familiar fact that the Higgs field is the differential of the period map. For the family of Section 3.1, the period map of $\mathcal{H}^1$ is not constant by Remark 4.11(ii), and Corollary 4.13 shows again that X contradicts Conjecture 1.1, in degrees 1 and 2.

Corollary 4.14. *In the situation of Theorem 4.9, assume that $\mathcal{X} \to U$ is an abelian scheme. Then X satisfies Conjecture 1.1 if and only if the Kodaira–Spencer map*

$$\mathrm{KS}: \quad f_*\Omega^1_{X/S} \longrightarrow \mathrm{R}^1f_*\mathcal{O}_X \otimes_{\mathcal{O}_S} \Omega^1_{S/K}$$

vanishes identically on S. If $\mathcal{X} \to U$ is a family of elliptic curves, this holds if and only if the j-invariant is constant on S.

Proof. Cup product gives isomorphisms $\wedge^i\mathcal{H}^1 \xrightarrow{\sim} \mathcal{H}^i$ compatible with the Gauss–Manin connections and, by E_1-degeneration, strictly compatible with the Hodge filtrations. Hence, by the Leibniz rule, θ_i is the derivation of $\wedge^i\mathcal{H}^1$ induced by θ_1. So $\theta_i|_S = 0$ for all i if and only if $\theta_1|_S = 0$. The only possibly nonzero component of θ_1 is $\theta^1_1 : f_*\Omega^1_{X/S} \to \mathrm{R}^1f_*\mathcal{O}_X \otimes \Omega^1_{S/K}$, which is KS (see the proof of Lemma 3.23). Theorem 4.9 allows us to conclude.

Suppose now that the fibers are elliptic curves. Étale locally on S, the family has Legendre form with parameter λ. By the formula in the proof of Lemma 3.23, KS = 0 if and only if $d\lambda = 0$, that is, if and only if λ is locally constant. Since the classical rational function $j = J(\lambda)$ has finite fibers, λ is locally constant if and only if j is locally constant. As S is connected, this is equivalent to j being constant on S. □

Remark 4.15. (i) The vanishing of the Kodaira–Spencer map is not a necessary condition for the total space of a family to satisfy Conjecture 1.1. For example, consider the whole algebraic Legendre family $f : \mathcal{E} \longrightarrow U = \mathbb{P}^1_{\mathbf{Q}_p} \smallsetminus \{0, 1, \infty\}$ of Remark 3.11. With the notations of Lemma 3.23, the same computation gives $\mathrm{KS}(\omega) = \frac{\overline{\eta}}{2\lambda(\lambda-1)} \otimes d\lambda$, which vanishes nowhere on U. On the other hand, as we have seen in Remark 3.11, the total space $\mathcal{E}^{\mathrm{an}}_C$ satisfies Conjecture 1.1.

(ii) The vanishing of the Higgs fields does not imply that $X \to S$ is isotrivial. For elliptic curves it does, by Corollary 4.14. But if the fibers satisfy $h^{a,b} = 0$ for $a \neq b$, as for smooth families of del Pezzo surfaces, all Higgs maps vanish automatically. Every such family, isotrivial or not, then satisfies Conjecture 1.1 over the discs of Hypothesis 4.5.

4.3. **Higher dimensional bases.** Let now U be a smooth rigid analytic variety of arbitrary dimension, $f : \mathcal{X} \to U$ smooth and proper, $S \subset U$ a dagger or open polydisc centered at $s_0 \in U(K)$, small enough for (4.6) to hold, and $X = \mathcal{X} \times_U S$. The period maps are defined as in Remark 4.11, and the first part of the proof of Lemma 4.7 and Section 4.2 still give, for all j and a, inclusions

$$F^aH^j_{\mathrm{dR}}(X) \;\subset\; H^0(S, F^a\mathcal{H}^j)^{\nabla=0} = \bigcap_{u\in S(K)} F^a_uV^j \;\subset\; F^a_sV^j, \tag{4.16}$$

and Conjecture 1.1 holds for X in degree i if and only if the composite inclusion is an equality for $j = i-1, i$ and all a. The second inclusion is an equality for all a if and only if the period map of $\mathcal{H}^j$ is constant, i.e., if and only if the Higgs maps θ^a_j are 0. The first one is an equality if and only if all horizontal sections of $F^a\mathcal{H}^j$ come from $H^j(X, \Omega^{\geq a}_X)$; this is automatic if S is a curve, by Lemma 4.2, which is why Corollary 4.13 holds, but it can fail if $\dim S \geq 2$. Similarly, the implications (i)⇒(iii)⇒(ii) of Theorem 4.9 still hold, with the same proofs, but not their converses, as the following example shows.

Let Y be a smooth projective curve of genus 2 over K, with $Y(K) \neq \varnothing$, let L_0 be a line bundle of degree $\neq 0$ on Y, and let $q \in K^\times$ with $v_p(q) > 0$. Let S be a small enough dagger polydisc centered at 0 in the Jacobian $\mathrm{Pic}^0(Y)$ of Y (in particular, S is of dimension 2), let $B := Y \times S$, let $\mathcal{L}$ be the restriction to B of $\mathrm{pr}_Y^* L_0 \otimes \mathcal{P}$, where $\mathcal{P}$ is the Poincaré bundle on $Y \times \mathrm{Pic}^0(Y)$, normalized along $\{y_0\} \times \mathrm{Pic}^0(Y)$ for some $y_0 \in Y(K)$, and let $\mathcal{L}^\times$ be the complement of the zero section in the total space of $\mathcal{L}$. Set

$$X := \mathcal{L}^\times / q^{\mathbf{Z}} \xrightarrow{\pi} B \xrightarrow{\mathrm{pr}_S} S, \qquad f := \mathrm{pr}_S \circ \pi.$$

Then f is smooth and proper, and its fibers X_s are principal bundles over Y under the Tate curve $E_q := \mathbf{G}_m / q^{\mathbf{Z}}$. We claim that all the Higgs maps of f are 0, i.e., that its period maps are constant, and that the local systems $\mathrm{R}^1 f_* \mathbf{Q}_p$ and $\mathrm{R}^2 f_* \mathbf{Q}_p$ are constant on S_C, but that $F^2 H^2_{\mathrm{dR}}(X) = 0$ whereas $\dim_K F^2 H^2_{\mathrm{dR}}(X_s) = 2$. So, for $i = 2$, conditions (ii) and (iii) of Theorem 4.9 and the condition of Corollary 4.13 hold, but the first inclusion in (4.16) is strict for $j = a = 2$, and X contradicts Conjecture 1.1 in degrees 2 and 3.

Let z be a local coordinate on the fibers of $\mathcal{L}^\times \to B$. Then $\Omega^1_{X/B} = \mathcal{O}_X \frac{dz}{z}$, and the class of the extension $0 \to \pi^* \Omega^1_B \to \Omega^1_X \to \mathcal{O}_X \frac{dz}{z} \to 0$ is the pullback of $c_1(\mathcal{L}) \in H^1(B, \Omega^1_B)$. Now,

$$c_1(\mathcal{L}) = c \otimes 1 + \tau_1 \otimes ds_1 + \tau_2 \otimes ds_2 \ \in \ \big(H^1(Y, \Omega^1_Y) \otimes_K \mathcal{O}(S)\big) \oplus \big(H^1(Y, \mathcal{O}_Y) \otimes_K \Omega^1(S)\big),$$

where $c = c_1(L_0) \neq 0$, and where τ_1, τ_2 is a basis of $H^1(Y, \mathcal{O}_Y)$ and s_1, s_2 are coordinates on S (the last two terms are the differential of the inclusion of S in $\mathrm{Pic}^0(Y)$, whose tangent space is $H^1(Y, \mathcal{O}_Y)$). The relative 2-forms on X/S are the $\alpha \wedge \frac{dz}{z}$, for $\alpha \in H^0(Y, \Omega^1_Y) \otimes_K \mathcal{O}(S)$, and such a form lifts to a 2-form on X if and only if there exists $\beta \in \Omega^1(S)$ such that $(\alpha + \beta) \cup c_1(\mathcal{L}) = 0$ in $H^1(B, \Omega^2_B)$. Writing $\alpha \cup \tau_k = \langle \alpha, \tau_k \rangle c$, this means

$$\langle \alpha, \tau_1 \rangle ds_1 + \langle \alpha, \tau_2 \rangle ds_2 = \pm \beta \quad \text{and} \quad \beta \wedge ds_1 = \beta \wedge ds_2 = 0. \tag{4.17}$$

• If we restrict f to a curve $S' \subset S$, the second condition is empty ($\Omega^2_{S'} = 0$) and the first defines β: all relative 2-forms lift, which means that the Higgs map θ_2^2 is 0 on S'. This being true for all S', we have $\theta_2^2 = 0$.

• Over S, the second condition implies $\beta = 0$, and then the first implies $\alpha = 0$, by Serre duality on Y. Hence all 2-forms on X are pullbacks of 2-forms on $B = Y \times S$, and their restrictions to the fibers X_s are 0 since Y is a curve. So $F^2 H^2_{\mathrm{dR}}(X) = 0$, whereas $F^2 H^2_{\mathrm{dR}}(X_s) = H^0(X_s, \Omega^2_{X_s}) \simeq H^0(Y, \Omega^1_Y)$ is of dimension 2.

• Since $c \neq 0$, the relative 1-forms on X/S are the pullbacks of the $\alpha \in H^0(Y, \Omega^1_Y) \otimes_K \mathcal{O}(S)$; they lift to X, hence $\theta_1^1 = 0$. The two remaining Higgs maps, θ_2^1 and θ_3^2, are the transposes of θ_2^2 and θ_1^1 for Serre duality.

• Since π is a principal bundle under E_q, the sheaves $\mathrm{R}^b \pi_* \mathbf{Q}_p$ are the constant sheaves $H^b_{\text{ét}}(E_{q,C}, \mathbf{Q}_p)$; hence the E_2-terms of the spectral sequence $\mathrm{R}^a \mathrm{pr}_{S*} \mathrm{R}^b \pi_* \mathbf{Q}_p \Rightarrow \mathrm{R}^{a+b} f_* \mathbf{Q}_p$ are the constant sheaves $H^a_{\text{ét}}(Y_C, \mathbf{Q}_p) \otimes H^b_{\text{ét}}(E_{q,C}, \mathbf{Q}_p)$, and its differentials d_2 are given by cup product with $c_1(L_0) \neq 0$. One deduces that $\mathrm{R}^2 f_* \mathbf{Q}_p \simeq H^1_{\text{ét}}(Y_C, \mathbf{Q}_p) \otimes H^1_{\text{ét}}(E_{q,C}, \mathbf{Q}_p)$, and that $\mathrm{R}^1 f_* \mathbf{Q}_p$ is an extension of $\mathbf{Q}_p$ by $H^1_{\text{ét}}(Y_C, \mathbf{Q}_p)$, split by the class of the $\mathbf{Z}$-covering $\mathcal{L}^\times \to X$. So both are constant.

In this example, the fibers X_s are p-adic analogues of non-Kähler elliptic surfaces: the same construction with $Y = \mathbf{P}^1$ and $L_0 = \mathcal{O}(-1)$ gives the Hopf surface $(\mathbf{A}^2 \setminus \{0\})/q^{\mathbf{Z}}$, and with Y of genus 1 a primary Kodaira surface. As for these, Hodge symmetry fails: $h^{1,0}(X_s) = 2$ and $h^{0,1}(X_s) = 3$. Both this failure and that of Conjecture 1.1 come from $c_1(L_0) \neq 0$, through the differential d_2 of the Leray spectral sequence of π. We do not know if the implication (ii)⇒(i) of Theorem 4.9 can also fail when f is projective.

Appendix A. De Rham filtrations

We compare the Hodge filtration with the induced filtration on $\mathbf{B}^+_{\mathrm{dR}}$-cohomology and describe the associated period lattices as intersections of fiber lattices.

A.1. **The Hodge filtration.** Our goal here is to prove the following result:

Proposition A.1. *If X is a smooth dagger variety defined over K with $H^*_{\rm dR}(X)$ finite dimensional, and if $n \le r$,*

$$\mathrm{Im}\big(H^n(F^r(\mathrm{R}\Gamma_{\rm dR}(X)\otimes^{\mathrm{L}\square}_K \mathbf{B}^+_{\rm dR})) \to H^n_{\rm dR}(X)\otimes_K \mathbf{B}^+_{\rm dR}\big) = F^r(H^n_{\rm dR}(X)\otimes_K \mathbf{B}^+_{\rm dR})$$

Proof. Let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{O}_C$ over $\mathcal{O}_K$ (i.e., the images modulo a uniformizer π_K of K form an algebraic basis of $\mathcal{O}_C/\pi_K\mathcal{O}_C$ over $\mathcal{O}_K/\pi_K$) and, if $i\in I$, choose $\hat{e}_i \in \mathbf{A}_{\inf,K} := \mathcal{O}_K\otimes\mathbf{A}_{\inf}$ such that $\theta(\hat{e}_i) = e_i$. Then $\mathbf{B}^+_{\rm dR}$ is, as a K-module[6], the set of

$$x = \sum_{k\ge 0}\big(\sum_{i\in I} x_{k,i}\hat{e}_i\big)t^k, \quad x_{k,i}\to 0 \text{ when } i\to\infty.$$

Moreover, this decomposition of x is unique, $x\mapsto x_{k,i}$ is K-linear and continuous[7], and $x\in F^r\mathbf{B}^+_{\rm dR}$ if and only if $x_{k,i}=0$ if $k<r$.

Now, let $\mathcal{U}=(U_j)$ be a covering of X by dagger affinoids. If J is a finite set of j's, let $U_J := \cap_{j\in J}U_j$. We write an element of $Z^n(\mathcal{U},\mathrm{R}\Gamma_{\rm dR}(X)\otimes^{\mathrm{L}\square}_K \mathbf{B}^+_{\rm dR})$ as $(\omega_J^{(n_J)})_J$, with $n_J = n-|J|+1$ and $\omega_J^{(n_J)}\in\Omega^{n_J}(U_J)\otimes^{\square}_K\mathbf{B}^+_{\rm dR}$.

The above decomposition of elements of $\mathbf{B}^+_{\rm dR}$ allows one to decompose each $\omega_J^{(n_J)}$ as

$$\omega_J^{(n_J)} = \sum_{k\ge 0}\big(\sum_{i\in I}\omega^{(n_J)}_{J,k,i}\hat{e}_i\big)t^k$$

and $\omega_J^{(n_J)}\mapsto\omega^{(n_J)}_{J,k,i}$ is K-linear and continuous, and commutes with differentiation and restriction to a sub-affinoid. It follows that each $(\omega^{(n_J)}_{J,k,i})_J$ is an n-cocycle belonging to $Z^n(\mathcal{U},\mathrm{R}\Gamma_{\rm dR}(X))$. Moreover, $(\omega_J^{(n_J)})_J\in Z^n(F^r(\mathrm{R}\Gamma_{\rm dR}(X)\otimes^{\mathrm{L}\square}_K\mathbf{B}^+_{\rm dR}))$ if and only if $\omega^{(n_J)}_{J,k,i}=0$ if $n_J+k<r$. Since $n_J+k<r$ is equivalent to $n_J<r-k$, this last condition is equivalent to $(\omega^{(n_J)}_{J,k,i})_J\in Z^n(\mathcal{U},\Omega^{\ge r-k})$ and to $t^k\hat{e}_i\otimes(\omega^{(n_J)}_{J,k,i})_J\in Z^n(\mathcal{U},\Omega^{\ge r-k})\otimes^{\mathrm{L}\square}_K t^k\mathbf{B}^+_{\rm dR}$. It follows that the image of $t^k\hat{e}_i\otimes(\omega^{(n_J)}_{J,k,i})_J$ in $H^n_{\rm dR}(X)\otimes_K\mathbf{B}^+_{\rm dR}$ belongs to $F^r(H^n_{\rm dR}(X)\otimes_K\mathbf{B}^+_{\rm dR})$, which proves the inclusion $\subset$ in the proposition since $F^r(H^n_{\rm dR}(X)\otimes_K\mathbf{B}^+_{\rm dR})$ is closed in $H^n_{\rm dR}(X)\otimes_K\mathbf{B}^+_{\rm dR}$.

In the other direction, if $x=a\otimes y$, with $a\in t^k\mathbf{B}^+_{\rm dR}$ and $y\in F^{r-k}H^n_{\rm dR}(X)$, one chooses a lifting $(\omega_J^{(n_J)})_J$ of y in $Z^n(\mathcal{U},\Omega^{\ge r-k})$, such that $\omega_J^{(n_J)}=0$ if $n_J<r-k$. Then

$$(a\omega_J^{(n_J)})_J\in Z^n(\mathcal{U},\Omega^{\ge r-k}\otimes^{\mathrm{L}\square}_K t^k\mathbf{B}^+_{\rm dR})\subset Z^n(\mathcal{U},F^r(\mathrm{R}\Gamma_{\rm dR}(X)\otimes^{\mathrm{L}\square}_K\mathbf{B}^+_{\rm dR})).$$

This proves the reverse inclusion, and finishes the proof. □

A.2. **$\mathbf{B}^+_{\rm dR}$-filtration.** Let S and X be as in Theorem 4.9, and let $V^i := H^0(S,\mathcal{H}^i)^{\nabla=0}$, so that $H^i_{\rm dR}(X)\simeq V^i$ (Lemma 4.7). If $u\in S(K)$, evaluation at u is an isomorphism $V^i\xrightarrow{\sim}H^i_{\rm dR}(X_u)$. We have a formula similar to (4.12) for the corresponding lattices: in $V^i\otimes_K\mathbf{B}_{\rm dR}$, we have

$$F^0(H^i_{\rm dR}(X)\otimes_K\mathbf{B}_{\rm dR}) = \bigcap_{u\in S(K)}F^0(H^i_{\rm dR}(X_u)\otimes_K\mathbf{B}_{\rm dR}). \tag{A.2}$$

(If $F^\bullet$ is a decreasing, separated and exhaustive filtration of a finite dimensional K-vector space V, then $F^0(V\otimes_K\mathbf{B}_{\rm dR}) = M(F^\bullet)\otimes_{K[t]}\mathbf{B}^+_{\rm dR}$, where $M(F^\bullet) := \oplus_{a\in\mathbf{Z}}t^{-a}F^aV\subset V\otimes_K K[t,t^{-1}]$. Now, $F^\bullet\mapsto M(F^\bullet)$ commutes with intersections, and so does $M\mapsto M\otimes_{K[t]}\mathbf{B}^+_{\rm dR}$ with finite intersections, because $\mathbf{B}^+_{\rm dR}$ is flat over $K[t]$. Hence (A.2) follows from (4.12).)

In other words, if we view the $\mathcal{E}^i(X_u)$ and $\mathcal{E}^i(X)$ as modifications at x_∞ of the vector bundle attached to $H^i_{\rm HK}(X_C)$, we have

$$\mathcal{E}^i(X) = \bigcap_{u\in S(K)}\mathcal{E}^i(X_u), \qquad \deg\mathcal{E}^i(X)-\deg\mathcal{E}^i(X_s) = -\sum_{a\ge 1}\big(\dim_K F^aH^i_{\rm dR}(X_s)-\dim_K F^aH^i_{\rm dR}(X)\big).$$

[6] See [1], Lemme 3.5 and Prop. 3.7, for similar constructions.

[7] We will not need this point, but it is also true that $x\mapsto\sum_{k\ge0}x_{k,i}t^k$ is $K[[t]]$-linear and continuous.

Here $s \in S(K)$ is arbitrary (the dimensions $\dim_K F^a H^i_{\mathrm{dR}}(X_s)$ are independent of s). Each of the $\mathcal{E}^i(X_u)$ is pure of slope 0 (hence also of degree 0), and their intersection is of slope 0 if and only if they are all equal: this is another way to phrase Lemma 4.8. Also, if $r \geq i$, we obtain, inside $\mathrm{HK}^{r,i}(X)$,

$$\mathrm{Ker}\,\pi^{r,i}(X) = \bigcap_{u\in S(K)} \mathrm{Ker}\,\pi^{r,i}(X_u) = \bigcap_{u\in S(K)} H^i_{\text{ét}}(X_{u,C}, \mathbf{Q}_p(r)).$$

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