ON THE PRO-KUMMER ÉTALE COHOMOLOGY OF $\mathbb{B}_{\mathrm{dR}}$

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Abstract. We investigate p-adic cohomologies of log rigid analytic varieties over a p-adic field. For a log rigid analytic variety X defined over a discretely valued field, we compute the Kummer pro-étale cohomology of $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathbb{B}_{\mathrm{dR}}$. When X is defined over $\mathbb{C}_p$, we introduce a logarithmic B_{dR}^+-cohomology theory, serving as a deformation of log de Rham cohomology. Additionally, we establish the log de Rham-étale comparison in this setting and prove the degeneration of both the Hodge-Tate and Hodge-log de Rham spectral sequences when X is proper and log smooth.

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1. Introduction

Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p. Let $W(k)$ be the ring of Witt vectors of k with fraction field F (therefore $W(k) = \mathscr{O}_F$). Let $\bar{K}$ be an algebraic closure of K and C be its p-adic completion, and let $\mathscr{O}_{\bar{K}}$ be the integer closure of $\mathscr{O}_K$ in $\bar{K}$. Let ϕ be the absolute Frobnius on $W(\bar{k})$. Set $\mathscr{G} = \mathrm{Gal}(\bar{K}/K)$.

We will denote by $\mathcal{O}_K$ and $\mathcal{O}_K^{\times}$, depending on the context, the scheme $\mathrm{Spec}(\mathcal{O}_K)$ or the formal scheme $\mathrm{Spf}(\mathcal{O}_K)$ with the trivial log structure and the canonical (i.e., associated to the closed point) log structure respectively.

1.1. **Background.** Before presenting the main theorem, we review some recent advancements in p-adic Hodge theory.

The aim of p-adic Hodge theory is to establish a Hodge-type decomposition over p-adic fields, analogous to the classical Hodge theory developed by Deligne and Hodge. For smooth algebraic varieties, such a decomposition was first achieved in [Fal88]. In the case of proper smooth rigid analytic varieties, Scholze extended this framework using perfectoid geometry, leading to the following results in [Sch13a].

Theorem 1.1. [Sch13a, Corollary 1.8] *For any proper smooth rigid-analytic variety X defined over K, the Hodge–de Rham spectral sequence*

$$E_1^{ij} := H^j(X, \Omega_X^i) \Rightarrow H_{\mathrm{dR}}^{i+j}(X)$$

degenerates at E_1, and there is a Hodge-Tate decomposition

$$H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_K C \simeq \bigoplus_{j=0}^{i} H^{i-j}(X, \Omega_X^j) \otimes_K C(-j).$$

Moreover, the p-adic étale cohomology $H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ is de Rham with associated filtered K-vector space $H^i_{\mathrm{dR}}(X)$.

Following Scholze's celebrated work on proper smooth rigid analytic varieties, numerous generalizations have emerged. The work by Hansheng Diao, Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu in [DLLZ23a] (see also the work of Shizhang Li and Xuanyu Pan in [LP19]) extended the above comparison theorem to almost proper rigid analytic varieties X, which means X can be written as $\overline{X} - Z$ with $\overline{X}$ being a proper smooth variety and Z a Zariski closed subset in $\overline{X}$. In [DLLZ23a], they further constructed the Simpson and Riemann-Hilbert correspondences for such varieties. Their approach involves developing the theory of log adic spaces, enabling a natural log structure on $\overline{X}$ induced by the open immersion $\overline{X} - Z \hookrightarrow \overline{X}$. By employing resolution of singularities, Z can be assumed to be a strictly normal crossing divisor on $\overline{X}$, allowing the reduction of the problem on X to one on $\overline{X}$, where methods akin to those in [Sch13a] apply.

Another direction of generalization involves exploring general rigid analytic varieties without requiring X to be proper or smooth. This was studied in [Bos23a] and [Bos23b]. In [Bos23a, Theorem 1.1.8], Guido Bosco computed the pro-étale cohomology of the period sheaf $\mathbb{B}_{\mathrm{dR}}$ for smooth rigid analytic varieties over a discretely valued field K. Within the framework of condensed mathematics, he computed the pro-étale cohomology of $\mathbb{B}_{\mathrm{dR}}$ via the Poincaré lemma, which, for proper smooth rigid analytic varieties over K, reproduces results from [Sch13a, Theorem 7.11]. In [Bos23b], by using éh-topology, Guido Bosco showed it is also possible to work with non-smooth rigid analytic varieties over an algebraically closed complete field C.

1.2. **The pro-Kummer étale cohomology of $\mathbb{B}_{\mathrm{dR}}$.** This article computes the pro-Kummer étale cohomology of $\mathbb{B}_{\mathrm{dR}}$ for log-smooth rigid analytic varieties over a discretely valued field K, extending the results of [Bos23a, Theorem 1.1.8].

In algebraic geometry, for a given smooth algebraic variety X, we typically consider a compactification $\overline{X}$ such that $\overline{X} - X$ is a strict normal crossing divisor. This setup allows $\overline{X}$ to carry a natural log structure that is log-smooth, thereby reducing many problems to the proper case where cohomology theory behaves more predictably. However, in rigid analytic geometry, compactifying a rigid analytic variety is generally not feasible (while it can be done in the category of adic spaces, the resulting space no longer remains a rigid analytic variety, introducing new complications). This is why, in [DLLZ23a], the authors restrict themselves considering almost proper rigid analytic varieties, which can be associated with a smooth proper rigid analytic variety endowed with a well-behaved log structure after resolution of singularities.

To extend the known comparison theorems for rigid analytic varieties, it is natural to introduce log structures on general (not necessarily proper) rigid analytic varieties. For pro-étale cohomology, recent works by Pierre Colmez, Gabriel Dospinescu, and Wiesława Nizioł [CN17], [CDN20], [CN20], [CN25] provide a detailed description of the pro-étale cohomology for smooth rigid analytic varieties, thanks to the comparison with syntomic cohomology. However, for non-quasi-compact rigid analytic varieties, computing étale cohomology remains challenging. By equipping a quasi-compact rigid analytic variety with a log structure derived from a strict normal crossing divisor, we can apply results on pro-étale cohomology to compute the étale cohomology of the trivial locus, which is itself not quasi-compact. Further exploration in log geometry, therefore, holds potential to clarify significant aspects of the étale cohomology of rigid analytic varieties.

The main theorem of this part is as follows (see Theorem 4.6 for a more general statement).

Theorem 1.2. *Let X be a log smooth rigid analytic variety defined over K with fine saturated log-structure. Define*

$$\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X_{B_{\mathrm{dR}}}) := \mathrm{R}\Gamma(X, \underline{\Omega_X^{\bullet,\log}} \otimes_K^{\blacksquare} \underline{B_{\mathrm{dR}}}).$$

(i) We have a $\mathscr{G}_K$-equivariant, compatible with filtrations, natural isomorphisms in $D(\mathrm{Mod}_K^{\mathrm{cond}})$:

$$\mathrm{R}\Gamma_{\underline{\mathrm{prok\acute{e}t}}}(X_C, \mathbb{B}_{\mathrm{dR}}) \simeq \mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X_{B_{\mathrm{dR}}}).$$

(ii) Assume X is connected and paracompact. Then, for each $r \in \mathbb{Z}$, we have a $\mathscr{G}_K$-equivariant natural quasi-isomorphism in $D(\mathrm{Mod}_K^{\mathrm{cond}})$:

$$\mathrm{R}\Gamma_{\underline{\mathrm{prok\acute{e}t}}}(X_C, \mathrm{Fil}^r\, \mathbb{B}_{\mathrm{dR}}) \simeq \mathrm{Fil}^r(\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X) \otimes_K^{L\blacksquare} \underline{B_{\mathrm{dR}}}),$$

where $\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X)$ is the condensed log de Rham cohomology complex in $D(\mathrm{Mod}_K^{\mathrm{cond}})$ (see Definition 2.1).

This result generalizes [Bos23a, Theorem 1.1.8]. By comparing these two results, we can relate the pro-Kummer étale cohomology and the pro-étale cohomology of $\mathbb{B}_{\mathrm{dR}}$ when the log-structure of X is coming from a strictly normal crossing divisor.

Corollary 1.3. *Let X be a smooth rigid analytic variety over K, $D \subset X$ be a strictly normal crossing divisor, and $U = X - D$. Endow X with the log-structure coming from D. Then we have a (non filtered!) natural quasi-isomorphism in $D(\mathrm{Mod}_K^{\mathrm{cond}})$:*

$$\mathrm{R}\Gamma_{\underline{\mathrm{prok\acute{e}t}}}(X_C, \mathbb{B}_{\mathrm{dR}}) \simeq \mathrm{R}\Gamma_{\underline{\mathrm{pro\acute{e}t}}}(U_C, \mathbb{B}_{\mathrm{dR}}).$$

It is unclear whether this corollary can be proved directly, as computations for Kummer (pro-)étale cohomology typically require case-by-case treatment. For instance, if X is quasi-compact and $D \subset X$ is a strictly normal crossing divisor, and $U = X - D$. We then have

$$\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X_C, \mathbb{Q}_p) \simeq \mathrm{R}\Gamma_{\underline{\text{két}}}(X_C, \mathbb{Q}_p) \simeq \mathrm{R}\Gamma_{\underline{\text{ét}}}(U_C, \mathbb{Q}_p),$$

which follows from [DLLZ23b, Corollary 6.3.4]. Note that $\mathrm{R}\Gamma_{\underline{\text{ét}}}(U_C, \mathbb{Q}_p) \neq \mathrm{R}\Gamma_{\underline{\text{proét}}}(U_C, \mathbb{Q}_p)$ in general. For example, when $U = \mathbb{A}^1$, $H^1_{\text{ét}}(A^1_C, \mathbb{Q}_p) = 0$ but $H^1_{\text{proét}}(A^1_C, \mathbb{Q}_p)$ is very large, as indicated by the main theorem of [CDN20].

1.3. **Geometric B^+_{dR}-cohomology and comparison theorem.** Now let X be a log smooth rigid analytic variety over C. Since the algebra B^+_{dR} lacks a natural C-structure, establishing a meaningful comparison theorem between the log de Rham cohomology and the Kummer étale cohomology necessitates the introduction of a new cohomology theory, $\mathrm{R}\Gamma_{B^+_{\mathrm{dR}}}(X)$. This theory serves as a deformation of log de Rham cohomology via the map $B^+_{\mathrm{dR}} \to C$.

For rigid analytic varieties without log structures, various approaches have been proposed to construct geometric B^+_{dR}-cohomology theories. For example:

- In [BMS18], Bhargav Bhatt, Matthew Morrow, and Peter Scholze introduce $\mathrm{R}\Gamma_{\mathrm{cris}}(X/B^+_{\mathrm{dR}})$ using the concept of "very small objects."
- In [Guo21], Haoyang Guo constructs $\mathrm{R}\Gamma(X/\Sigma_{\mathrm{inf}}, \mathcal{O}_{X/\Sigma_{\mathrm{inf}}})$ by studying the infinitesimal site over B^+_{dR}.
- In [CN25], Pierre Colmez and Wiesława Nizioł define $\mathrm{R}\Gamma_{\mathrm{dR}}(X/B^+_{\mathrm{dR}})$ via local semistable formal models of X.

Notably, all these definitions are shown to be equivalent by [CN25, Proposition 3.27]. In [Bos23b], by using the $L\eta_f$-functor, Guido Bosco introduces $\mathrm{R}\Gamma_{B^+_{\mathrm{dR}}}(X)$ via period sheaves, and proves that it agrees with the geometric B^+_{dR}-cohomology.

In this article, we adopt Bosco's approach to define $\mathrm{R}\Gamma_{B^+_{\mathrm{dR}}}(X)$, as we believe it provides the most direct framework for defining logarithmic B^+_{dR}-cohomology with minimal prerequisites. After establishing such a theory, we will prove the following comparison theorem.

Theorem 1.4. *Let X be a proper log smooth rigid analytic varieties over C. Then there are cohomology groups $H^i_{B^+_{\mathrm{dR}}}(X)$ which come with a canonical filtered isomorphism*

$$H^i_{\text{két}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq H^i_{B^+_{\mathrm{dR}}}(X) \otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}.$$

When X comes from X_0 over a discretely valued field K, this isomorphism agrees with the comparison isomorphism (see [DLLZ23a, Theorem 1.1]*)*

$$H^i_{\text{két}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq H^i_{\mathrm{logdR}}(X_0) \otimes_K B_{\mathrm{dR}},$$

under a canonical identification

$$H^i_{B^+_{\mathrm{dR}}}(X) = H^i_{\underline{\mathrm{logdR}}}(X_0) \otimes_K B^+_{\mathrm{dR}}.$$

Moreover, $H^i_{\mathrm{logdR}}(X/B^+_{\mathrm{dR}})$ is a finite free B^+_{dR}-module, and we have

(i) The Hodge–de Rham spectral sequence

$$E_1^{ij} := H^j(X, \Omega_X^{\log,i}) \Rightarrow H^{i+j}_{\mathrm{logdR}}(X)$$

degenerates at E_1.

(ii) The Hodge–Tate spectral sequence

$$E_2^{ij} := H^j(X, \Omega_X^{\log,i})(-j) \Rightarrow H^{i+j}_{\text{két}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$$

degenerates at E_2.

These results serve as an initial step toward the study of rational p-adic Hodge theory for logarithmic rigid analytic varieties. In fact, we can define the syntomic cohomology for logarithmic rigid analytic varieties by adopting a similar framework. We will explore these ideas in detail in a forthcoming paper [Sha25b].

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Notation and conventions. In this article, We freely use the language of condensed mathematics as developed in [CS19]. To avoid set-theoretic issues, we fix an uncountable strong limit cardinal κ, which means κ is uncountable and for all $\lambda < \kappa$, also $2^\lambda < \kappa$. For sheaves valued in condensed abelian groups, we refer the reader to [Bos23a] for notations and properties used in this article. In particular, here are some notations that are used in this article:

- We denote by CondAb the category of condensed abelian groups. For a discretely valued field K, we denote by $\mathrm{Mod}_K^{\mathrm{cond}}$ and $\mathrm{Mod}_K^{\mathrm{solid}}$ the category of condensed K-vector spaces and the category of solid K-vector spaces, respectively.
- For a topological abelian group T, we can associate T with a condensed abelian group $\underline{T}$ as follows: for any profinite group S, we define $\underline{T}(S) := \mathscr{C}(S, T)$ the set of continuous maps from S to T. Conversely, for a condensed abelian group A, we denote by $A(*)$ the underlying abelian group of A.
- The above construction allows us to relate a sheaf $\mathcal{F}$ of topological abelian groups over a topological space X with a sheaf $\underline{\mathcal{F}}$, by sending any open $U \subset X$ to $\underline{\mathcal{F}}(U) := \underline{\mathcal{F}(U)}$.
- For $\tau =$ ét or proét, and an analytic adic space X, we can define the site $X_{\tau,\mathrm{cond}}$ to retaining the information captured by profinite sets, see [Bos23a, 2.3].

We adopt the language of adic spaces as developed in [Hub96]. A rigid analytic variety is defined as a quasi-separated adic space locally of finite type over $\mathrm{Spa}(L, \mathcal{O}_L)$ for a p-adic field L. All rigid analytic spaces considered will be over K or C. We assume all rigid analytic varieties are separated, taut, and countable at infinity. We denote by Sm_L the category of smooth rigid analytic varieties (or smooth dagger varieties) over L. Moreover, we always assume analytic adic spaces are κ-small, meaning the cardinality of their underlying topological spaces is less than κ.

If $\mathcal{A}$ is an abelian category, unless stated otherwise, we always work with derived stable ∞-category $D(\mathcal{A})$.

We also use the theory of log adic space. For definitions and properties of log adic space, we refer the reader to [DLLZ23b]. A (pre) log-structure on a condensed ring is simply a (pre)log-structure on the underlying ring.

2. Logarithmic de Rham cohomology

In this section, we study the de Rham cohomology for logarithmic rigid analytic varieties. We will show that most of the known results continue to hold in the condensed mathematics setting.

Suppose X is a log smooth rigid analytic varieties over $L = K$ or C, i.e. an analytic fs log adic space which is locally of finite type over $\mathrm{Spa}(L, \mathcal{O}_L)$. Recall that in [DLLZ23b, Construction 3.3.2], one has logarithmic differential étale sheaf $\Omega_X^{1,\log}$ (also denote by $\Omega_X^{\log}$), locally defined by the universal objects of derivations. When X is log smooth, by [DLLZ23b, Lemma 3.3.15] $\Omega_X^{1,\log}$ is locally free of finite rank. For any $i \geq 1$, let $\Omega_X^{i,\log} := \bigwedge^i \Omega_X^{1,\log}$.

For any affinoid étale morphism $U \to X$, $\Omega_X^{i,\log}(U)$ is a finite $\mathcal{O}_U(U)$-module, therefore it has a natural structure of K-Banach space. This leads to the following definition.

Definition 2.1. Suppose X is a log smooth rigid analytic varieties over L, the logarithmic de Rham cohomology of X is defined to be

$$\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X) := \mathrm{R}\Gamma\left(X, \underline{\Omega_X^{\bullet,\log}}\right)$$

in $D(\mathrm{Mod}_K^{\mathrm{cond}})$.

Recall that the usual de Rham cohomology for log rigid analytic varieties is defined by

$$\mathrm{R}\Gamma_{\mathrm{logdR}}(X) := \mathrm{R}\Gamma\left(X, \Omega_X^{\bullet,\log}\right).$$

When X is log smooth and proper, $H^i_{\mathrm{logdR}}(X)$ is finite dimensional and carries a natural topology, therefore the structure of $\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X)$ is simple.

Lemma 2.2. *Suppose X is log smooth and proper, then we have*

$$\underline{H^i_{\mathrm{logdR}}(X)} = H^i_{\underline{\mathrm{logdR}}}(X)$$

for all $i \geq 0$.

Proof. The proof of [Bos23a, Lemma 5.11] goes through. □

We are mainly interested in the case where the log structure of X comes from a strictly normal crossing divisor $D \subset X$. Following [Gro66], [Kie67] and [Del70], we give the following definition.

Definition 2.3. Suppose X is a smooth rigid analytic variety over L, $D \subset X$ is a strictly normal crossing divisor, and $U = X - D$. Let $i \geq 1$.

(i) We define $\Omega^i_X(*D)$ to be the differential i-forms on X with meromorphic poles along D. In other words, let $\mathscr{I}$ be the ideal sheaf associated to the closed immersion $D \hookrightarrow X$, then

$$\Omega^i_X(*D) := \varinjlim_n \mathcal{H}om_{\mathcal{O}_X}(\mathscr{I}^n, \Omega^i_X).$$

(ii) We define $\Omega^i_X(\log D)$ to be the differential i-forms on X with logarithmic poles along D. In other words, $\Omega^i_X(\log D)$ contains the elements ω of $\Omega^i_X(*D)$ such that ω and $\mathrm{d}\omega$ have a pole of order at most 1.

We can also give a local description of the structure of $\Omega^i_X(\log D)$. By the existence of tubular neighborhood, [Kie67, Theorem 1.18], locally under the analytic topology X can be written as

$V := \mathrm{Sp}(A\langle x_1, ..., x_d\rangle)$, and D is defined by the equation $x_1 \cdots x_d = 0$, where A is a smooth affinoid L-algebra. Therefore, $\Omega^i_X(\log D)(V)$ is generated by forms

$$\omega \cdot (x_{i_1} \cdots x_{i_q})^{-1} dx_{i_1} \cdots dx_{i_q}$$

for $\omega \in \Omega^{i-q}_A$ and $1 \leq i_1 < \cdots < i_q \leq d$.

Lemma 2.4. *We have natural isomorphisms*

$$\Omega^{i,\log}_X \xrightarrow{\simeq} \Omega^i_X(\log D),$$

for each $i \geq 1$, where the log-structure of X is given by the open immersion $U \hookrightarrow X$.

Proof. We can check the lemma locally. By base change, we may assume $V = \mathrm{Sp}(L\langle x_1, ..., x_d\rangle)$ as above, and D is defined by the equation $x_1 \cdots x_d = 0$, and the log-structure of V is defined by the open immersion $V - D \hookrightarrow V$. We need to construct a natural isomorphism

$$\Omega^{1,\log}_X(V) \xrightarrow{\simeq} \Omega^1_X(\log D)(V).$$

According to [DLLZ23b, Proposition 3.2.25] (taking $P = \varnothing$ and $Q = \mathbb{Z}^d_{\geq 0} = \bigoplus_{i=1}^d \mathbb{Z}_{\geq 0} e_i$), we have

$$\Omega^{1,\log}_X(V) \simeq \bigoplus_{i=1}^d L\langle x_1, ..., x_d\rangle e_i.$$

We can then construct the natural isomorphism $\Omega^{1,\log}_X(V) \xrightarrow{\simeq} \Omega^1_X(\log D)(V)$ by sending e_i to $x_i^{-1} dx_i$. This concludes the proof, as the construction of the above map is independent of the representation of the tubular neighborhood. $\square$

Now we endow X with the log-structure defined by the open immersion $U \hookrightarrow X$. Since X is log smooth (e.g. see [DLLZ23b, Example 3.1.13]), $\Omega^i_X(\log D)$ is a coherent sheaf, therefore has a natural structure of sheaves of K-Banach spaces, and we have a natural condensed sheaf of $\mathcal{O}_X$-modules $\underline{\Omega^i_X(\log D)}$.

Choose a basis $\mathcal{B}$ of X_{an} consisting of objects of the form $V = \mathrm{Sp}(A\langle x_1, ..., x_d\rangle)$ as above, and endow

$$\Omega^i_X(*D)(V) \simeq \varinjlim_n \mathcal{H}om_{\mathcal{O}_X}(\mathscr{I}^n, \Omega^i_X)(V)$$

with the direct limit topology. Then $\Omega^i_X(*D)$ is a sheaf of topological abelian groups on $\mathcal{B}$. We denote by $\underline{\Omega^i_X(*D)}$ the condensed sheaf associated to $\Omega^i_X(*D)$.

Lemma 2.5. *We have*

$$\underline{\Omega^i_X(*D)} \simeq \varinjlim_n \underline{\mathcal{H}om_{\mathcal{O}_X}(\mathscr{I}^n, \Omega^i_X)}.$$

Remark 2.6. This lemma does not hold automatically, as the condensification functor does not commute with colimits in general.

Proof. Denote by

$$\mathcal{F}_n := \mathcal{H}om_{\mathcal{O}_X}(\mathscr{I}^n, \Omega^i_X).$$

By definition, we need to show the natural morphism

$$\varinjlim_n \underline{\mathcal{F}_n} \to \underline{\varinjlim_n \mathcal{F}_n} = \underline{\Omega^i_X(*D)}$$

is an isomorphism. To see this, by [BS15, Lemma 4.3.7], it suffices to show for a basis $\mathcal{B}$ of X consisting of objects of the form $V = \mathrm{Sp}(A\langle x_1, ..., x_d\rangle)$ as above, $\mathcal{F}_n(V) \to \mathcal{F}_{n+1}(V)$ is a closed immersion. Since the sheaf of ideal $\mathscr{J}$ is generated by $x_1 \cdots x_d = 0$, and Ω^i_X is locally free of finite rank, by shrinking V (refining $\mathcal{B}$) and base change we are reduced to showing the map of topological abelian groups

$$L\langle x_1, ..., x_d\rangle \xrightarrow{\cdot x_1 \cdots x_d} L\langle x_1, ..., x_d\rangle$$

is a closed immersion, which is easy to check: in fact, if $f \in L\langle x_1, ..., x_d\rangle$ is not in the ideal $(x_1 \cdots x_d)$, then for N suffices large, $f + p^N \mathcal{O}_L\langle x_1, ..., x_d\rangle \cap (x_1 \cdots x_d) = \varnothing$. ☐

Remark 2.7. In fact, denote the open immersion $U \hookrightarrow X$ by j, it is easy to see that we have inclusions

$$\Omega^i_X(\log D)(V) \hookrightarrow \Omega^i_X(*D)(V) \hookrightarrow j_*\Omega^i_U(V)$$

of topological abelian groups, by checking their sections.

We want to compare the hypercohomology of $\Omega^\bullet_X(\log D)$ and the de Rham cohomology of U. This leads to the following theorem.

Theorem 2.8. [Kie67, Theorem 2.3] *Suppose X is a smooth rigid analytic varieties over L, $D \subset X$ is a strictly normal crossing divisor, and $U = X - D$. Then we have (non-filtered) quasi-isomorphisms in $D(\mathrm{Mod}^{\mathrm{cond}}_L)$:*

$$\mathrm{R}\Gamma(X, \underline{\Omega^\bullet_X(\log D)}) \xrightarrow{\simeq} \mathrm{R}\Gamma(X, \underline{\Omega^\bullet_X(*D)}) \xrightarrow{\simeq} \mathrm{R}\Gamma_{\underline{\mathrm{dR}}}(U).$$

Proof. Take $V = \mathrm{Sp}(A\langle x_1, ..., x_d\rangle)$ as above, define

$$V^\circ := \{v \in V \text{ such that } |x_1(v)| < 1, ..., |x_d(v)| < 1\}.$$

Consider the natural maps

$$\Omega^\bullet_X(\log D)(V^\circ) \xrightarrow{s} \Omega^\bullet_X(*D)(V^\circ) \xrightarrow{r} j_*\Omega^\bullet_U(V^\circ),$$

where j is the open immersion $j : U \hookrightarrow X$. In [Kie67, Theorem 2.3], Kiehl constructed maps $\mathrm{Res} : j_*\Omega^\bullet_U(V^\circ) \to \Omega^\bullet_X(\log D)(V^\circ)$ and $\partial : j_*\Omega^\bullet_U(V^\circ) \to j_*\Omega^\bullet_U(V^\circ)$, and showed that $r \circ s$ and $\mathrm{Res} \circ r$ are both homotopy equivalences given by ∂. Therefore, to show the same is true for condensed cohomology groups, it is enough for us to check that s, r, ∂ and Res are continuous, as we can lift continuous maps between topological groups to their condensifications in a natural way. According to the previous remark, it suffices to show that the maps Res and ∂ are continuous, which follows from the explicitly construction of [Kie67, Theorem 2.3]. This concludes the proof. ☐

This theorem also allows us to describe the logarithmic de Rham cohomology for Stein varieties with log-structures. We begin with the following lemma.

Lemma 2.9. [GK04, Corollary 3.2] *Let X be a smooth Stein variety over L, $D \subset X$ is a strictly normal crossing divisor, let $U = X - D$. Endow X with the compactifying log-structure given by the open immersion $U \hookrightarrow X$. Then for each $i \geq 0$, the image of the differential $d : \Omega^{i-1,\log}_X(X) \to \Omega^{i,\log}_X(X)$ is a closed sub-space of closed images $\Omega^{i,\log}_X(X)$.*

Proof. The proof of [GK04, Corollary 3.2] goes through by using Theorem 2.8, [GK04, Theorem 3.1], and the fact that $\Omega^{i,\log}_X$ are coherent sheaves over X for all $i \geq 0$. ☐

With the notations in the above lemma, for X a smooth Stein variety, endowing $\Omega^{i,\log}(X)^{d=0} \subset \Omega^{i,\log}(X)$ with the subspace topology, and $H^i_{\mathrm{logdR}}(X) = \Omega^{i,\log}(X)^{d=0}/d\Omega^{i-1,\log}(X)$ with the induced quotient topology, we have

Proposition 2.10. [Bos23a, Lemma 5.13] *With the above notations, for all $i \geq 0$, we have*

$$\underline{H^i_{\mathrm{logdR}}}(X) = \underline{H^i_{\mathrm{logdR}}(X)} = \underline{\Omega^{i,\log}(X)^{d=0}/d\Omega^{i-1,\log}(X)}.$$

Proof. Since $\Omega^{i,\log}(X)^{d=0}, d\Omega^{i-1,\log}(X)$ and $H^i_{\mathrm{logdR}}(X)$ are all L-Fréchet spaces, this proposition follows from [Bos23a, Lemma 5.9] and [Bos23a, Lemma A.33]. $\square$

3. Period sheaves

We review the definition of period sheaves for log adic space and their local properties, which follow from [DLLZ23a].

Suppose X is an analytic fs log adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

3.1. Definitions.

We begin with the basic definition.

Definition 3.1. The following are defined to be sheaves on $X_{\text{prokét}}$ (see [DLLZ23b, Definition 5.1.2] for the definition):

(i) We define $\mathbb{A}_{\inf} := W(\widehat{\mathcal{O}}^+_{X^\flat_{\text{prokét}}})$ and $\mathbb{B}_{\inf} := \mathbb{A}_{\inf}[1/p]$, where the latter is equipped with a natural map $\theta : \mathbb{B}_{\inf} \to \widehat{\mathcal{O}}_{X_{\text{prokét}}}$.

(ii) We define the positive de Rham sheaf $\mathbb{B}^+_{\mathrm{dR}} := \lim_{n\in\mathbb{N}} B_{\inf,X}/(\ker\theta)^n$ with filtration given by $\mathrm{Fil}^r\,\mathbb{B}^+_{\mathrm{dR}} := (\ker\theta)^r\mathbb{B}^+_{\mathrm{dR}}$.

(iii) We define the de Rham sheaf $\mathbb{B}_{\mathrm{dR}} := \mathbb{B}^+_{\mathrm{dR}}[1/t]$, where t is a generator of $\mathrm{Fil}^1\,\mathbb{B}^+_{\mathrm{dR}}$. The filtration of $\mathbb{B}_{\mathrm{dR}}$ is given by $\mathrm{Fil}^r\,\mathbb{B}_{\mathrm{dR}} := \sum_{j\in\mathbb{Z}, r+j\geq 0} t^{-j}(\ker\theta)^{r+j}\mathbb{B}^+_{\mathrm{dR}}$.

The following proposition shows that pro-Kummer étale locally these period sheaves are given by the relative period rings.

Proposition 3.2. [DLLZ23a, Proposition 2.2.4] *Suppose $U \in X_{\text{prokét}}$ is a log affinoid perfectoid object with associated perfectoid space $\widehat{U} = \mathrm{Spa}(R, R^+)$ (see [DLLZ23b, Remark 5.3.5] for the definition), then*

(i) For $\mathcal{F} \in \{\widehat{\mathcal{O}}^+, \widehat{\mathcal{O}}^{\flat,+}, \mathbb{A}_{\inf}, \mathbb{B}_{\inf}, \mathbb{B}^+_{\mathrm{dR}}, \mathbb{B}_{\mathrm{dR}}\}$, we have $\mathcal{F}(U) = \mathcal{F}(R, R^+)$.

(ii) $H^j(U, \mathbb{B}^+_{\mathrm{dR}}) = 0$ and $H^j(U, \mathbb{B}_{\mathrm{dR}}) = 0$ for $j > 0$.

All these period rings carry a natural condensed sheaf structure defined as follows: For a given X an analytic log adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$ and $\mathcal{F} \in \{\widehat{\mathcal{O}}^+, \widehat{\mathcal{O}}^{\flat,+}, \mathbb{A}_{\inf}, \mathbb{B}_{\inf}, \mathbb{B}^+_{\mathrm{dR}}, \mathbb{B}_{\mathrm{dR}}\}$, denote $\mathcal{B}$ the basis of $X_{\text{prokét}}$ consisting of log affinoid perfectoid objects $U \in X_{\text{prokét}}$, then for any $U \in \mathcal{B}$, $\mathcal{F}(U)$ carry a natural structure of topological abelian groups, by giving $\widehat{\mathcal{O}}^+(U)$ the p-adic topology, and endowing all other period sheaves with induced topology, as explained in [Sch13a, Corollary 6.6]. We can then associate to $\mathcal{F}$ a condensed abelian sheaf $\underline{\mathcal{F}}$ on $X_{\text{prokét}}$, such that $\underline{\mathcal{F}}(U) = \underline{\mathcal{F}(U)}$ for all log affinoid perfectoid objects $U \in \mathcal{B}$.

We pass now to condensed pro-Kummer étale cohomology. We begin with the definition of condensed pro-Kummer étale cohomology.

Definition 3.3. Let X be an analytic log adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Denote by $*_{\kappa-\text{proét}}$ the site of κ-small profinite sets with coverings given by finite families of jointly surjective maps. There is a natural morphism of sites

$$f_{\text{cond}} : X_{\text{prokét}} \to *_{\kappa-\text{proét}}$$

defined by sending a profinite set S to $X \times S \in X_{\text{prokét}}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $X_{\text{prokét}}$, the condensed pro-Kummer étale cohomology of $\mathcal{F}$ is then defined by

$$\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathcal{F}) := Rf_{\text{cond},*}\mathcal{F}.$$

Denote by $H^i_{\underline{\text{prokét}}}(X, \mathcal{F}) := R^i f_{\text{cond},*}\mathcal{F}$ its i-th cohomology group valued in CondAb.

Remark 3.4. When X is an analytic log adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$, denote by $f : X_{\text{prokét}} \to \mathrm{Spa}(C, \mathcal{O}_C)_{\text{proét}}$ the morphism of sites, then the above definition coincides with the derived pushforward of f, i.e. we have $\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathcal{F}) = Rf_*\mathcal{F}$.

Proposition 3.5. *Let X be an analytic log adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$ and $\mathcal{F} \in \{\widehat{\mathcal{O}}^+, \widehat{\mathcal{O}}^{\flat,+}, \mathbb{A}_{\text{inf}}, \mathbb{B}_{\text{inf}}, \mathbb{B}^+_{\text{dR}}, \mathbb{B}_{\text{dR}}\}$. For any log affinoid perfectoid object $U \in \mathcal{B}$ and profinite set $S \in *_{\kappa-\text{proét}}$, we have*

$$\mathcal{F}(U \times S) = \mathscr{C}^0(S, \mathcal{F}(U)).$$

In particular, we have

$$H^i_{\underline{\text{prokét}}}(U, \mathcal{F}) = H^i_{\text{prokét}}(U, \underline{\mathcal{F}})$$

for all $i \geq 0$.

Proof. The proof of [Sch13a, Corollary 6.6] or [Bos23a, Corollary 4.9] goes through. □

Now let X be a locally noetherian fs log adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$. We recall the definition of $\mathcal{O}\mathbb{B}_{\text{dR,log}}$, a log version of the geometric de Rham period sheaf $\mathcal{O}\mathbb{B}_{\text{dR}}$. Those notations and properties will be used in the following sections.

Let $U = \lim_{i \in I} U_i \in X_{\text{prokét}}$ be a log affinoid perfectoid object with $U_i = (\mathrm{Spa}(R_i, R_i^+), \mathcal{M}_i, \alpha_i)$ for each $i \in I$ and the associated perfectoid space $\widehat{U} = \mathrm{Spa}(R_\infty, R_\infty^+)$, where (R_∞, R_∞^+) is the p-adic completion of $\mathrm{colim}_{i \in I}(R_i, R_i^+)$, which is perfectoid. For each $i \in I$, write $M_i := \mathcal{M}_i(U_i)$. By [DLLZ23b, Theorem 5.4.3], $(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U)) = (R_\infty, R_\infty^+)$, and the tilt of $(\widehat{\mathcal{O}}(U), \widehat{\mathcal{O}}^+(U))$ is $(R_\infty^{\flat+}, R_\infty^\flat)$. Define $M := \mathcal{M}_{X_{\text{prokét}}}(U) = \lim_{i \in I} M_i$, $M^\flat := \mathcal{M}_{X^\flat_{\text{prokét}}}(U) = \lim_{a \mapsto a^p} M$, and $\alpha^\flat : M^\flat \to \widehat{\mathcal{O}}^\flat$ the induced map.

Definition 3.6. (i) The period sheaf $\mathcal{O}\mathbb{B}^+_{\text{dR,log}}$ is defined to be the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{i \in I} \left(R_i^+ \widehat{\otimes}_{W(k)} \mathbb{A}_{\text{inf}}(U)\right)[1/p]\left[\frac{\alpha_i(m_i)}{[\alpha^\flat(m^\flat)]}, (m_i, m_i^\flat) \in M_i \times_M M^\flat\right]^{\wedge_{\ker \theta_{\log}}}.$$

Here $\widehat{\otimes}$ is the p-adic completion of the tensor product, $[\alpha^\flat(m^\flat)]$ is obtained by the multiplicative map

$$R^{+\flat} \to W(R^{+\flat})[1/p]/\xi^r : f \mapsto [f]$$

induced by $R^{+\flat} \to W(R^{+\flat})$, for each $r \geq 1$, and

$$\theta_{\log} : \left(R_i^+ \widehat{\otimes}_{W(k)} \mathbb{A}_{\text{inf}}(U)\right)[1/p]\left[\frac{\alpha_i(m_i)}{[\alpha^\flat(m^\flat)]}, (m_i, m_i^\flat) \in M_i \times_M M^\flat\right] \to R_\infty$$

is induced by the map $R_i^+ \to R_\infty^+$, $\theta : A_{\mathrm{inf}}(U) \to R_\infty^+$ such that $\theta_{\log}\left(\dfrac{\alpha_i(m_i)}{[\alpha^\flat(m^\flat)]}\right) = 1$ for any $(m_i, m_i^\flat) \in M_i \times_M M^\flat$. In above, "$\wedge_{\ker\theta_{\log}}$" means the completion with respect to the $\ker\theta_{\log}$-adic topology, and we take the direct limit after the completion. We equip $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+$ with the filtration $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+ := (\ker\theta_{\log})^r \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+$.

(ii) The period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ is the completion of the sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+[1/t]$ with respect to the filtration defined by $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+[1/t] := \sum_{j\in\mathbb{Z}, r+j\geq 0} t^{-j}(\ker\theta_{\log})^{r+j}\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+$.

There is a natural log connection

$$\nabla : \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+ \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega_X^{\log}$$

on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ which is defined as follows: By abuse of notation, we still denote by $\Omega_X^{\log}$ the pullback of log differential sheaf $\Omega_X^{\log}$ along the morphism of sites $X_{\text{prokét}} \to X_{\text{ét}}$, and we may write $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+(U)$ as

$$\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+(U) = \varinjlim_{i\in I} \widehat{S}_i = \varinjlim_{i\in I}\varprojlim_{r,s}(S_{i,r}/(\ker\theta_{\log})^s).$$

Here

$$S_{i,r} = \left(R_i^+ \widehat{\otimes}_{W(k)}(\mathbb{A}_{\mathrm{inf}}(U))/\xi^r\right)[1/p]\left[\frac{\alpha_i(m_i)}{[\alpha^\flat(m^\flat)]}, (m_i, m_i^\flat) \in M_i \times_M M^\flat\right],$$

and $\widehat{S}_i := \varprojlim_{r,s}(S_{i,r}/(\ker\theta_{\log})^s)$. Then there is a unique $\mathbb{B}_{\mathrm{dR}}^+(U)/\xi^r$-linear log connection

$$\nabla_{i,r} : S_{i,r} \to S_{i,r} \otimes_{R_i} \Omega_X^{\log}(U_i)$$

extending derivations $d : R_i \to \Omega_X^{\log}(U_i)$ and $\delta : M_i \to \Omega_X^{\log}(U_i)$ such that

$$\nabla_{i,r}\left(\frac{\alpha_i(m_i)}{[\alpha^\flat(m^\flat)]}\right) = \frac{\alpha_i(m_i)}{[\alpha^\flat(m^\flat)]}\delta(a)$$

for any $(m_i, m_i^\flat) \in M_i \times_M M^\flat$. Then the definition of $\nabla_{i,r}$ gives

$$\nabla_{i,r}\left((\ker\theta_{\log})^s\right) \subset (\ker\theta_{\log})^{s-1} \otimes_{R_i} \Omega_X^{\log}(U_i)$$

for all $s \geq 1$. Then the $\mathbb{B}_{\mathrm{dR}}^+$-linear log connection

$$\nabla : \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+ \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega_X^{\log}$$

is obtained by taking

$$\nabla := \varinjlim_{i\in I}\varprojlim_{r}\left(\nabla_{i,r}^{\wedge_{\ker\theta_{\log}}}\right).$$

By inverting t, ∇ further extends to a $\mathbb{B}_{\mathrm{dR}}$-linear log connection

$$\nabla : \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \to \mathcal{O}\mathbb{B}_{\mathrm{dR},\log} \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega_X^{\log},$$

satisfying

$$\nabla\left(\mathrm{Fil}^r\, O\mathbb{B}_{\mathrm{dR},\log}\right) \subset \left(\mathrm{Fil}^{r-1}\, O\mathbb{B}_{\mathrm{dR},\log}\right) \otimes_{\mathcal{O}_{X_{\text{prokét}}}} \Omega_X^{\log}$$

for all $r \in \mathbb{Z}$.

3.2. **Local study of $\mathbb{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$.** We briefly discuss the local properties of $\mathbb{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$, as studied in [DLLZ23a, 2.3]. Let P be a toric monoid (i.e. a fs sharp monoid). Denote by

$$\mathbb{E} := \mathrm{Spa}(K\langle P\rangle, \mathcal{O}_K\langle P\rangle)$$

with the log-structure induced by the natural map $P \to K\langle P\rangle$. The log adic space $\mathbb{E}$ admits a pro-Kummer étale cover as follows. For each $m \in \mathbb{Z}_{>0}$, let $\frac{1}{m}P$ be the toric monoid such that $P \hookrightarrow \frac{1}{m}P$ can be identified with the multiple map $[m] : P \to P$. Let $P_{\mathbb{Q}_{\geq 0}} := \mathrm{colim}_m \left(\frac{1}{m}P\right)$ and $P^{\mathrm{gp}}_{\mathbb{Q}} := \left(P_{\mathbb{Q}_{\geq 0}}\right)^{\mathrm{gp}} \simeq P^{\mathrm{gp}} \otimes \mathbb{Q}$. Denote by

$$\mathbb{E}_{m,C} := \mathrm{Spa}\left(C\left\langle\frac{1}{m}P\right\rangle, \mathcal{O}_C\left\langle\frac{1}{m}P\right\rangle\right)$$

equipped with the log-structure induced by $\frac{1}{m}P \to K\langle\frac{1}{m}P\rangle$. The morphism $\mathbb{E}_m \to \mathbb{E}$ is a finite Kummer étale cover with Galois group

$$\Gamma_m = \mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{\mu}_m),$$

where $\boldsymbol{\mu}_m$ is the group of the m-th roots of unity in C. Denote the log affinoid perfectoid object by

$$\widetilde{\mathbb{E}}_C := \varprojlim_m \mathbb{E}_{m,C} \in \mathbb{E}_{C,\text{prokét}},$$

where for $m|m'$, the transition maps are induced by $\frac{1}{m}P \hookrightarrow \frac{1}{m'}P$. Then we have the associated perfectoid space

$$\widehat{\widetilde{\mathbb{E}}}_C := \mathrm{Spa}(C\langle P_{\mathbb{Q}_{\geq 0}}\rangle, \mathcal{O}_C\langle P_{\mathbb{Q}_{\geq 0}}\rangle).$$

The morphism $\widehat{\widetilde{\mathbb{E}}}_C \to \mathbb{E}_C$ is a Galois pro-finite Kummer étale cover with Galois group

$$\Gamma = \mathrm{Hom}(P^{\mathrm{gp}}_{\mathbb{Q}}/P^{\mathrm{gp}}, \boldsymbol{\mu}_\infty),$$

where $\boldsymbol{\mu}_\infty := \cup_{m\in\mathbb{N}}\boldsymbol{\mu}_m$. The natural action of the Galois group Γ on $\mathcal{O}_C\langle P_{\mathbb{Q}_{\geq 0}}\rangle$ is given by

$$\gamma(T^a) = \gamma(a)T^a$$

for all $\gamma \in \Gamma$ and $a \in P_{\mathbb{Q}_{\geq 0}}$, where T^a is the corresponding element of a in $\mathcal{O}_C\langle P_{\mathbb{Q}_{\geq 0}}\rangle$.

Suppose $X = \mathrm{Spa}(R, R^+)$ is an affinoid fs log adic space of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$ with a strictly étale morphism $X \to \mathbb{E}$, which can be written as a composite of rational embedding and finite étale maps. Pulling back $\widehat{\widetilde{\mathbb{E}}}_C \to \mathbb{E}$ along $X \to \mathbb{E}$, we get

$$\widetilde{X}_C := X_C \times_{\mathbb{E}_C} \widehat{\widetilde{\mathbb{E}}}_C = \mathrm{Spa}(R_\infty, R^+_\infty),$$

which is a Galois pro-finite Kummer étale cover with Galois group Γ.

Let $\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P]]$ be the sheaf of monoid algebras. For $a \in P$, denote by e^a the image of a in $\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P]]$ via the natural map $P \to \mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P]] : a \mapsto e^a$. Let $\mathfrak{m} \subset \mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[P]$ be the sheaf of ideals generated by $\{e^a - 1\}_{a\in P}$, and denote by

$$\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P-1]] := \varprojlim_r(\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P]]/\mathfrak{m}^r).$$

Consider the monoid homomorphism (with respect to the additive structure on $\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P-1]]$)

$$P \to \mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P-1]] : a \mapsto \log(e^a) := \sum_{l=1}^{\infty} (-1)^{l-1} \frac{1}{l} (e^a - 1)^l,$$

which uniquely extends to a group homomorphism

$$P^{\mathrm{gp}} \to \mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P-1]] : a \mapsto y_a := \log(e^{a^+}) - \log(e^{a^-}),$$

where $a = a^+ - a^-$ for $a^+, a^- \in P$. Choose a $\mathbb{Z}$-basis $\{a_1, ..., a_n\}$ of P^{gp}, and for each $j = 1, 2, ..., n$, denote by $y_j := y_{a_j}$, then we have a canonical isomorphism of $\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}$-algebras:

$$\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[y_1, ..., y_n]] \xrightarrow{\simeq} \mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P-1]] : y_j \mapsto y_j := y_{a_j},$$

matching the ideals $(y_1, .., y_n)^r$ and $(\xi, y_1, .., y_n)^r$ of the source with the ideals $\mathfrak{m}^r$ and $(\xi, \mathfrak{m})^r$ of the target, respectively, for all $r \in \mathbb{Z}$.

Now, similar to [Sch13a, Proposition 6.10], we can give a local description of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$. Recall that we denote by $U = \varprojlim_{i \in I} U_i \in X_{\text{prokét}}/\widetilde{X}_C$ a log affinoid perfectoid object with $U_i = (\mathrm{Spa}(R_i, R_i^+), \mathcal{M}_i, \alpha_i)$. Denote by $\alpha^\flat : \mathcal{M}^\flat := \lim_{a \mapsto a^p} \mathcal{M} \to \widehat{\mathcal{O}}$ the induced map. Consider the map (note that the sheaf $\mathcal{M}^\flat|_{\widetilde{X}_C}$ is generated by $P_{\mathbb{Q}_{\geq 0}}$ and $\varprojlim_{f \mapsto f^p} \mathcal{O}^\times_{X_{\text{prokét}}}|_{\widetilde{X}_C}$, therefore $\alpha^\flat(a)$ makes sense)

$$(\mathbb{B}^+_{\mathrm{dR}}(U)/\xi^r)[P] \to S_{i,r} : e^a \mapsto \frac{\alpha(a)}{[\alpha^\flat(a)]}, \text{ for all } a \in P,$$

which sends $(\xi, \mathfrak{m})$ to $\ker(\theta_{\log})$, so this map induces a map $\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[P-1]] \to \widehat{S}_i$. After taking completion and sheafification, we obtain a map

$$\mathbb{B}^+_{\mathrm{dR}}|_{\widetilde{X}}[[P-1]] \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}|_{\widetilde{X}} \tag{3.1}$$

on $X_{\text{prokét}}/\widetilde{X}_C$. The map is compatible with filtrations on both sides, where the filtration on $\mathbb{B}^+_{\mathrm{dR}}|_{\widetilde{X}_C}[[P-1]]$ is given by $\mathrm{Fil}^r\, \mathbb{B}^+_{\mathrm{dR}}|_{\widetilde{X}_C}[[P-1]] := (\xi, \mathfrak{m})^r \mathbb{B}^+_{\mathrm{dR}}|_{\widetilde{X}_C}[[P-1]]$ for all $r \in \mathbb{Z}$. We have the following proposition.

Proposition 3.7. [DLLZ23a, Proposition 2.3.15] *The map (3.1) is an isomorphism of filtered sheaves.*

Remark 3.8. In fact, for $U = \varprojlim_{i \in I} U_i$ as above, the natural map $\widehat{S}_i \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}(U)$ is already an isomorphism.

Remark 3.9. We can also describe the log connection of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$ using this isomorphism. In fact, choose a $\mathbb{Z}$-basis $\{a_1, ..., a_n\}$ of P^{gp}, write $a_j = a_j^+ - a_j^-$ for $j = 1, 2, ..., n$, then $y_j = b_j^+ - b_j^- := \log\left(e^{a_j^+}\right) - \log\left(e^{a_j^-}\right)$, and the isomorphism (3.1) sends y_j to $\dfrac{\alpha(b_j^+)}{[\alpha^\flat(b_j^+)]} - \dfrac{\alpha(b_j^-)}{[\alpha^\flat(b_j^-)]}$. Therefore,

$$\nabla(y_j) = \nabla\left(\frac{\alpha(b_j^+)}{[\alpha^\flat(b_j^+)]}\right) - \nabla\left(\frac{\alpha(b_j^-)}{[\alpha^\flat(b_j^-)]}\right) = \delta(a_j^+) - \delta(a_j^-) = \delta(a_j).$$

We note that, over $\widetilde{X}$, the log connection of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$ is compatible with the canonical one of the polynomial algebra $\mathbb{B}_{\mathrm{dR}}|_{\widetilde{X}_C}[[y_1, ..., y_n]]$, since $\Omega^{\log}_X \simeq \bigoplus_{j=1}^n \mathcal{O}_X \delta(a_j)$ by [DLLZ23b, Theorem 3.3.17, Corollary 3.3.18, Proposition 3.2.25, and Corollary 3.2.29].

Corollary 3.10. [DLLZ23b, Corollary 2.3.17] *The isomorphism 3.1 induces isomorphisms*

$$\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}} \simeq t^r\mathbb{B}^+_{\mathrm{dR}}\{W_1, ..., W_n\}$$

over $X_{\text{prokét}}/\widehat{X}$, for all $r \in \mathbb{Z}$. Here $\mathbb{B}^+_{\mathrm{dR}}\{W_1, ..., W_n\}$ is the ring of power series that are t-adically convergent, and

$$W_j = t^{-1}y_j$$

for each $1 \le j \le n$. In particular,

$$\mathrm{gr}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR,log}} \simeq t^r\widehat{\mathcal{O}}_{X_{\text{prokét}}}[W_1, ..., W_n],$$

for all $r \in \mathbb{Z}$.

The above discussion allows us to deduce the Poincaré lemma for $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$ with log connections.

Proposition 3.11. [DLLZ23a, Corollary 2.4.2] *Let X be a log smooth rigid analytic variety with a fine log-structure defined over K of dimension n. Then, we have an exact sequence of sheaves on $X_{\text{prokét}}$:*

$$0 \to \mathbb{B}^+_{\mathrm{dR}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}} \otimes_{\mathcal{O}_X} \Omega^{\log,1}_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}} \otimes_{\mathcal{O}_X} \Omega^{\log,n}_X \to 0.$$

The exact sequence of sheaves also holds when we replace $\mathbb{B}^+_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$ with $\mathbb{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$ respectively. For $r \in \mathbb{Z}$ we also have compatible exact sequences of sheaves on $X_{\text{prokét}}$:

$$0 \to \mathrm{Fil}^r\,\mathbb{B}_{\mathrm{dR}} \to \mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR,log}} \xrightarrow{\nabla} (\mathrm{Fil}^{r-1}\,\mathcal{O}\mathbb{B}_{\mathrm{dR,log}})\otimes_{\mathcal{O}_X}\Omega^{\log,1}_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} (\mathrm{Fil}^{r-n}\,\mathcal{O}\mathbb{B}_{\mathrm{dR,log}})\otimes_{\mathcal{O}_X}\Omega^{\log,n}_X \to 0.$$

4. The pro-Kummer étale cohomology of $\mathbb{B}^+_{\mathrm{dR}}$ and $\mathbb{B}_{\mathrm{dR}}$

We will establish an analog version of [Bos23a, Theorem 1.8] for log adic spaces. Moreover, we will also construct a fully faithful functor from the category of filtered $\mathcal{O}_X$-modules with integrable connections to the category of $\mathbb{B}_{\mathrm{dR}}$-local systems, thereby generalizing [Sch13a, Theorem 7.6].

We start with the definition of local systems for log adic spaces. Let X be a log smooth rigid analytic variety over K, equipped with a fine log structure.

Definition 4.1. (i) A $\mathbb{B}^+_{\mathrm{dR}}$-local system is a sheaf of $\mathbb{B}^+_{\mathrm{dR}}$-module $\mathbb{M}^+$ that is locally on $X_{\text{prokét}}$ free of finite rank.

(ii) An $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$-module with integrable log connection is a sheaf of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$-module $\mathcal{M}$ that is locally on $X_{\text{prokét}}$ free of finite rank, together with an integrable log connection

$$\nabla_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^{\log}_X,$$

satisfying the Leibniz rule with respect to the derivation ∇ of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$.

Theorem 4.2. *The functor*

$$\mathbb{M}^+ \mapsto (\mathcal{M}, \nabla_{\mathcal{M}}) := \left(\mathbb{M}^+ \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}, \mathrm{id} \otimes \nabla\right)$$

gives an equivalence between the category of $\mathbb{B}^+_{\mathrm{dR}}$-local systems and the category of $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$-modules with integrable log connection. The inverse is given by $(\mathcal{M}, \nabla_{\mathcal{M}}) \mapsto \mathbb{M}^+ := (\mathcal{M})^{\nabla_{\mathcal{M}}=0}$.

Proof. It is clear that for a $\mathbb{B}^+_{\mathrm{dR}}$-local system $\mathbb{M}^+$,

$$\left(\mathbb{M}^+ \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}\right)^{\nabla=0} = \mathbb{M}^+.$$

We need to show that for $(\mathcal{M}, \nabla_M)$ a $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$-module with integrable log connection, the natural morphism

$$(\mathcal{M})^{\nabla_{\mathcal{M}}=0} \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}} \xrightarrow{\simeq} \mathcal{M},$$

is an isomorphism. The problem is local, so we can use Proposition 3.7 and reduce to the case that $X = \mathrm{Spa}(R, R^+) \to \mathbb{E}$ is a strictly étale morphism, which can be written as a composite of rational embedding and finite étale maps. Now suppose M is a locally free $\mathbb{B}^+_{\mathrm{dR}}(R^\infty)[[y_1, ..., y_n]]$-module with an integrable log connection ∇_M, using Remark 3.9 and [Kat70, Proposition 8.9], we have $M \simeq M^{\nabla=0} \otimes_{\mathbb{B}^+_{\mathrm{dR}}(R^\infty)} \mathbb{B}^+_{\mathrm{dR}}(R^\infty)[[y_1, ..., y_n]]$, which concludes the proof. □

Definition 4.3. [Sch13a, Definition 7.4 and Definition 7.5] (i) A filtered $\mathcal{O}_X$-module with integrable log connection is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ on X (with étale, pro-étale or analytic topology, since they are all equivalent), together with a separated and exhaustive decreasing filtration $\{\mathrm{Fil}^r \mathcal{E}\}_{r\in\mathbb{Z}}$, by locally direct summands, and an integrable log connection ∇ satisfying the Griffiths transversality.

(ii) We say $\mathcal{E}$ and an $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$-module with integrable log connection $\mathcal{M}$ are associated if there is an isomorphism of sheaves on $X_{\text{prokét}}$

$$\mathcal{M} \otimes_{\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}} \simeq \mathcal{E} \otimes_{\mathcal{O}_X} O\mathbb{B}_{\mathrm{dR,log}},$$

compatible with filtrations and connections, where the filtration on the left side is the one compatible with that on $O\mathbb{B}_{\mathrm{dR,log}}$, i.e.,

$$\mathrm{Fil}^r(\mathcal{M} \otimes_{O\mathbb{B}^+_{\mathrm{dR,log}}} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}) := \sum_{j\in\mathbb{Z}, r+j\geq 0} t^{-j}(\ker \theta_{\log})^{r+j}\mathcal{M}.$$

Theorem 4.4. *(i) If $\mathcal{M}$ is an $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$-module with integrable log connection and horizontal section $\mathbb{M}^+$, which is associated to a filtered $\mathcal{O}_X$-module with integrable connection $\mathcal{E}$, then*

$$\mathbb{M}^+ = \mathrm{Fil}^0(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}})^{\nabla=0}.$$

Similarly, one can reconstruct $\mathcal{E}$ by

$$\mathcal{E}_{\text{ét}} \simeq \nu_* \left(\mathbb{M}^+ \otimes_{\mathbb{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}\right),$$

with log connection induced by the one on $\mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$, filtration induced by the one on $\mathbb{M}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$. Here ν is the morphism of site $X_{\text{prokét}} \to X_{\text{ét}}$.

(ii) If $\mathcal{E}$ is an $\mathcal{O}_X$-module with integrable log connection, the sheaf

$$\mathbb{M}^+ = \mathrm{Fil}^0(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}})^{\nabla=0}$$

is a $\mathbb{B}_{\mathrm{dR}}^{+}$*-local system such that* $\mathcal{E}$ *is associated to* $\mathcal{M} = \mathbb{M}^{+} \otimes_{\mathbb{B}_{\mathrm{dR}}^{+}} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$.

In particular, the functor sending $\mathcal{E}$ *to* $\mathbb{M}^{+}$ *is a fully faithful functor from the category of filtered* $\mathcal{O}_X$*-module with integrable log connection to the category of* $\mathbb{B}_{\mathrm{dR}}^{+}$*-local systems.*

Proof. The proof goes as the same way as the proof of [Sch13a, Theorem 7.6], by replacing $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ with $\mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$ and Ω_X with $\Omega_X^{\log}$. □

Definition 4.5. Let X be a log smooth rigid-analytic variety over K with fine log-structure. Let $(\mathcal{E}, \nabla, \mathrm{Fil}^{\bullet})$ be a filtered $\mathcal{O}_X$-module with integrable log connection. We define the log de Rham complex associated to $(\mathcal{E}, \nabla)$ by

$$\mathrm{logdR}_X^{\mathcal{E}} := \left[\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^{\log,1} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^{\log,2} \xrightarrow{\nabla} \cdots\right].$$

We equip $\mathrm{logdR}_X^{\mathcal{E}}$ with the filtration given by

$$\mathrm{Fil}^r \mathrm{logdR}_X^{\mathcal{E}} := \left[\mathrm{Fil}^r \mathcal{E} \xrightarrow{\nabla} \mathrm{Fil}^{r-1} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^{\log,1} \xrightarrow{\nabla} \mathrm{Fil}^{r-2} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^{\log,2} \xrightarrow{\nabla} \cdots\right]$$

for $r \in \mathbb{Z}$.

(i) For $i \geq 0$, the condensed de Rham cohomology group $H^i_{\underline{\mathrm{logdR}}}(X, \mathcal{E})$ with coefficient is defined to be the i-th cohomology group of the complex

$$\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X, \mathcal{E}) := \mathrm{R}\Gamma(X, \underline{\mathrm{logdR}_X^{\mathcal{E}}})$$

of $D(\mathrm{Mod}_K^{\mathrm{cond}})$.

(ii) We define the complex in $D(\mathrm{Mod}_K^{\mathrm{cond}})$

$$\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X_{B_{\mathrm{dR}}}, \mathcal{E}) := \mathrm{R}\Gamma(X, \underline{\mathrm{logdR}_X^{\mathcal{E}}} \otimes_K^{\blacksquare} \underline{B_{\mathrm{dR}}}),$$

and we endow it with the filtration induced from the tensor product filtration.

We will prove the following theorem.

Theorem 4.6. *Let* X *be a log smooth rigid analytic variety defined over* K *with fine saturated log-structure. Let* $(\mathcal{E}, \nabla, \mathrm{Fil}^{\bullet})$ *be a filtered* $\mathcal{O}_X$*-module with integrable log connection with associated* $\mathbb{B}_{\mathrm{dR}}^{+}$*-local system* $\mathbb{M}^{+}$*. Denote by* $\mathbb{M} := \mathbb{M}^{+}[1/t]$.

(i) We have a $\mathscr{G}_K$*-equivariant, compatible with filtrations, natural quasi-isomorphism in* $D(\mathrm{Mod}_K^{\mathrm{cond}})$*:*

$$\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X_C, \mathbb{M}) \simeq \mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X_{B_{\mathrm{dR}}}, \mathcal{E}).$$

(ii) Assume X *is connected and paracompact. Then, for each* $r \in \mathbb{Z}$*, we have a natural* $\mathscr{G}_K$*-equivariant quasi-isomorphisms in* $D(\mathrm{Mod}_K^{\mathrm{cond}})$*:*

$$\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X_C, \mathrm{Fil}^r \mathbb{M}) \simeq \mathrm{Fil}^r(\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X, \mathcal{E}) \otimes_K^{\blacksquare} \underline{B_{\mathrm{dR}}}),$$

where $\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X)$ *is the log de Rham cohomology complex in* $D(\mathrm{Mod}_K^{\mathrm{cond}})$*, and the filtration is induced by the tensor product filtration.*

To prove the theorem, we need the following proposition, which comes from the local study of $\mathbb{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$, as in Section 3 and we follow the notations there. Suppose that $X = \mathrm{Spa}(R, R^{+})$ is an affinoid fs log adic space of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$ with a strictly étale morphism $X \to \mathbb{E}$, which can be written as a composite of rational embeddings and finite étale maps. Pulling back $\widehat{\widetilde{\mathbb{E}}}_C \to \mathbb{E}_C$ along $X \to \mathbb{E}$, we get

$$\widetilde{X}_C := X_C \times_{\mathbb{E}_C} \widehat{\widetilde{\mathbb{E}}}_C = \mathrm{Spa}(R_\infty, R_\infty^+),$$

which is a perfectoid log affinoid adic space.

Proposition 4.7. [Bos23a, Lemma 6.11] *Suppose $S \in *_{\kappa,\text{proét}}$, then we have*

$$H^i_{\text{prokét}}(X_C \times S, \mathrm{gr}^j\, \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}) = \begin{cases} \mathscr{C}^0(S, R\widehat{\otimes}_K C(j)) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases},$$

where $C(j)$ is the Tate twist of degree j for any $j \in \mathbb{Z}$.

Proof. By twisting we can reduce to the case $j = 0$. When $S = *$, this is [DLLZ23a, Lemma 3.3.15]. For the general case, since $\widehat{\widetilde{\mathbb{E}}}_C \to \mathbb{E}_C$ is a Galois pro-finite Kummer étale cover with Galois group Γ, the Cartan–Leray spectral sequence associated to the affinoid perfectoid Γ-cover $\widetilde{X}_C \times S \to X_C \times S$, combined with the vanishing theorem for $\mathrm{gr}^0\, \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}$ (by Proposition 3.2 and Proposition 3.5) gives

$$H^i_{\text{prokét}}(X_C \times S, \mathrm{gr}^0\, \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}) = H^i_{\mathrm{cont}}(\Gamma, \mathrm{gr}^0\, \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}(X_C \times S))$$

for $i \geq 0$. Note that $X_C \times S = \mathrm{Spa}(\mathscr{C}^0(S, R_\infty), \mathscr{C}^0(S, R_\infty^+))$. The proofs of [LZ17] and [DLLZ23a, Lemma 3.3.15] give $H^i_{\text{prokét}}(X_C \times S, \mathrm{gr}^0\, \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}) = 0$ for $i > 0$ and

$$H^0_{\text{prokét}}(X_C \times S, \mathrm{gr}^0\, \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}) = \mathscr{C}^0(S, R)\widehat{\otimes}_K C.$$

The proposition then follows from the fact that $\mathscr{C}^0(S, R)\widehat{\otimes}_K C \simeq \mathscr{C}^0(S, R\widehat{\otimes}_K C)$, which follows from [PGS10, Corollary 10.5.4]. ☐

Proposition 4.8. *Let X be a log smooth rigid analytic variety defined over K with fine log-structure. Let $(\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)$ be a filtered $\mathcal{O}_X$-module with integrable log connection with associated $\mathbb{B}^+_{\mathrm{dR}}$-local system $\mathbb{M}^+ := \mathrm{Fil}^0(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}})^{\nabla=0}$, and $\mathbb{M} = \mathbb{M}^+[1/t]$. Denote by λ the morphism of sites*

$$X_{\text{prokét}}/X_C \simeq X_{C,\text{prokét}} \to X_{C,\text{ét},\text{cond}}.$$

Then, we have a natural quasi-isomorphism of complexes of sheaves valued in $D(\mathrm{Mod}_K^{\mathrm{cond}})$ on $X_{C,\text{ét}}$:

$$(R\lambda_*\mathbb{M})^\blacktriangledown \simeq \underline{\mathrm{logdR}_X^{\mathcal{E}}} \otimes_K^\blacksquare \underline{B_{\mathrm{dR}}},$$

where the notation $(-)^\blacktriangledown$ is define in [Bos23a, 2.3]. Moreover, the quasi-isomorphism is compatible with filtrations, where the filtration on the right side is given by the tensor product filtration.

Proof. Suppose X is of dimension n and connected. Denote by $\mathcal{M}$ the $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$-module with integrable log connection associated to $(\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)$, and

$$\mathcal{M}' := \mathcal{M} \otimes_{O\mathbb{B}^+_{\mathrm{dR,log}}} O\mathbb{B}_{\mathrm{dR,log}} \simeq \mathcal{E} \otimes_{\mathcal{O}_X} O\mathbb{B}_{\mathrm{dR,log}}.$$

The logarithmic Poincaré lemma gives an exact sequence of sheaves on $X_{C,\text{prokét}}$:

$$0 \to \mathbb{M} \to \mathcal{M}' \xrightarrow{\nabla} \mathcal{M}' \otimes_{\mathcal{O}_X} \Omega_X^{\log,1} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{M}' \otimes_{\mathcal{O}_X} \Omega_X^{\log,n} \to 0,$$

which is also exact after taking Fil^r again by the logarithmic Poincaré lemma:

$$0 \to \mathrm{Fil}^r \mathbb{M}' \to \mathrm{Fil}^r \mathcal{M}' \xrightarrow{\nabla} (\mathrm{Fil}^{r-1} \mathcal{M}') \otimes_{\mathcal{O}_X} \Omega_X^{\log,1} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} (\mathrm{Fil}^{r-n} \mathcal{M}') \otimes_{\mathcal{O}_X} \Omega_X^{\log,n} \to 0.$$

Therefore, we have a natural quasi-isomorphism between $(R\lambda_* \mathbb{M})^\blacktriangledown$ and

$$\mathcal{C}^\bullet := R\lambda_* \left[\mathcal{M}' \xrightarrow{\nabla} \mathcal{M}' \otimes_{\mathcal{O}_X} \Omega_X^{\log,1} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{M}' \otimes_{\mathcal{O}_X} \Omega_X^{\log,n}\right]^\blacktriangledown,$$

which is compatible with filtrations.

We claim we have a natural filtered quasi-isomorphism

$$\underline{\mathrm{logdR}_X^{\mathcal{E}}} \otimes_K^\blacksquare \underline{B_{\mathrm{dR}}} \xrightarrow{\simeq} \mathcal{C}^\bullet \tag{4.1}$$

of complexes of sheaves on $X_{C,\text{ét}}$ (as explained in the Lemma 2.5 of [LZ17], we can regard them as sheaves over $X_{C,\text{ét}}$). To construct this morphism, it suffices to define a natural morphism $\mathcal{O}_X \otimes_K^\blacksquare \underline{B_{\mathrm{dR}}} \to (\lambda_* \mathcal{O}\mathbb{B}_{\mathrm{dR},\log})^\blacktriangledown$ of sheaves on $X_{C,\text{ét}}$, which is compatible with filtration. This is already constructed in the proof of [Bos23a, Corollary 6.12] (also in [LZ17, Lemma 3.7] and [DLLZ23a, Lemma 3.3.2]), by noting that the pushforward of $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ along the natural morphism of sites $X_{C,\text{prokét}} \to X_{C,\text{proét}}$ is $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$.

As explained in [Bos23a, Corollary 6.12], since the filtrations on $\underline{\mathrm{logdR}_X^{\mathcal{E}}} \otimes_K^\blacksquare \underline{B_{\mathrm{dR}}}$ and $\mathcal{C}^\bullet$ are exhaustive and complete, by [BMS19, Lemma 5.2], it suffices to show the natural morphism (4.1) is a quasi-isomorphism on graded pieces, which means that for any $r \in \mathbb{Z}$, we have a natural isomorphism

$$\mathrm{gr}^j(\underline{\mathrm{logdR}_X^{\mathcal{E}}} \otimes_K^\blacksquare \underline{B_{\mathrm{dR}}}) \to \mathrm{gr}^j \mathcal{C}^\bullet$$

on $X_{C,\text{ét}}$. Since (4.1) is a filtered morphism, it suffices to show that for any locally free $\mathcal{O}_X$-module $\mathcal{F}$ on $X_{\text{ét}}$ of finite rank, we have a natural quasi-isomorphism

$$\underline{\mathcal{F}} \otimes_K^\blacksquare \mathrm{gr}^i \underline{B_{\mathrm{dR}}} \xrightarrow{\simeq} R\lambda_* \left(\nu^* \mathcal{F} \otimes_{\mathcal{O}_X} \mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}\right)^\blacktriangledown$$

on $X_{C,\text{ét}}$, for any $r \in \mathbb{Z}$, where $\nu : X_{\text{prokét}} \to X_{\text{ét}}$. Then we can apply the claim to the complex of sheaves $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^{\log,\bullet}$. Write $\lambda : X_{C,\text{prokét}} \to X_{C,\text{ét},\text{cond}}$ as

$$X_{C,\text{prokét}} \xrightarrow{\alpha} X_{C,\text{proét}} \xrightarrow{\lambda'} X_{C,\text{ét},\text{cond}},$$

and write ν as

$$X_{\text{prokét}} \to X_{\text{proét}} \xrightarrow{\nu'} X_{\text{ét}}.$$

According to the proof of [Bos23a, Corollary 6.12], we have a natural quasi-isomorphism

$$\underline{\mathcal{F}} \otimes_K^\blacksquare \mathrm{gr}^j \underline{B_{\mathrm{dR}}} \xrightarrow{\simeq} R\lambda'_* \left(\nu'^* \mathcal{F} \otimes_{\mathcal{O}_X} \mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{dR}}\right)^\blacktriangledown$$

on $X_{C,\text{ét}}$, for any $j \in \mathbb{Z}$. Therefore, it suffices to show that we have a natural quasi-isomorphism

$$R\lambda'_* \left(\nu'^* \mathcal{F} \otimes_{\mathcal{O}_X} \mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{dR}}\right) \xrightarrow{\simeq} R\lambda_* \left(\nu^* \mathcal{F} \otimes_{\mathcal{O}_X} \mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}\right)$$

on $X_{C,\text{proét}}$, for any $j \in \mathbb{Z}$. Since $\mathcal{F}$ is locally free, it suffices to show that

$$R\lambda_* \left(\mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{dR}}\right) \to R\lambda_* \mathrm{gr}^j \mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$$

is an isomorphism for any $j \in \mathbb{Z}$, which follows from [LZ17, Lemma 3.7] and [DLLZ23a, Lemma 3.3.2]. $\square$

Now we can prove Theorem 4.6.

Proof of theorem 4.6. The morphism of sites $\lambda : X_{C,\text{prokét}} \to X_{C,\text{ét},\text{cond}}$ factors as

$$\lambda : X_{C,\text{prokét}} \xrightarrow{\mu} X_{C,\text{prokét},\text{cond}} \to X_{C,\text{ét},\text{cond}}.$$

We have

$$\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X_C, \mathbb{M}) \simeq \mathrm{R}\Gamma_{\text{prokét},\text{cond}}(X_C, R\mu_*\mathbb{M}) \simeq \mathrm{R}\Gamma_{\text{ét},\text{cond}}(X_C, R\lambda_*\mathbb{M}) \simeq \mathrm{R}\Gamma_{\text{ét}}(X_C, (R\lambda_*\mathbb{M})^{\blacktriangledown}).$$

Let $\epsilon : X_{C,\text{ét}} \to X_{\text{ét}}$ be the base change morphism, then the above property, combined with [Bos23a, Lemma 5.6] and [Bos23a, Lemma 6.13], gives

$$\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X_C, \mathbb{M}) \simeq \mathrm{R}\Gamma_{\text{ét}}(X, R\epsilon_* \left(R\lambda_*\mathbb{M}\right)^{\blacktriangledown}) \simeq \mathrm{R}\Gamma_{\text{ét}}(X, \underline{\mathrm{logdR}}^{\mathcal{E}}_X \otimes^{\blacksquare}_K \underline{B_{\mathrm{dR}}}),$$

which conclude the proof of (i).

For the second part, it suffices to show that the natural morphism

$$\mathrm{Fil}^r(\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X, \mathcal{E}) \otimes^{L\blacksquare}_K B_{\mathrm{dR}}) \to \mathrm{R}\Gamma(X, \mathrm{Fil}^r(X_C, \underline{\mathrm{logdR}}^{\mathcal{E}}_X \otimes^{\blacksquare}_K \underline{B_{\mathrm{dR}}})) \simeq \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X_C, \mathrm{Fil}^r \mathbb{M})$$

is a quasi-isomorphism. This follows from [Bos23a, Theorem 5.20], by applying it to each $\mathrm{Fil}^i \mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X, \mathcal{E}) \otimes^{L\blacksquare}_K \mathrm{Fil}^j B_{\mathrm{dR}}$ for $i + j = r$, and noting that $\mathrm{R}\Gamma(X, -)$ commutes with filtered colimits. □

Remark 4.9. If X is a quasi-compact smooth rigid analytic varieties over K, we have a $\mathscr{G}_K$-equivariant, compatible with filtrations, natural isomorphismsin $D(\mathrm{Mod}^{\mathrm{cond}}_K)$:

$$\begin{aligned}
\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X_C, \mathbb{M}) &\simeq \mathrm{colim}_{j\in\mathbb{N}} \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X_C, \mathrm{Fil}^{-j} \mathbb{M}) \\
&\simeq \mathrm{colim}_{j\in\mathbb{N}} \mathrm{Fil}^{-j}(\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X, \mathcal{E}) \otimes^{L\blacksquare}_K B_{\mathrm{dR}}) \\
&\simeq \mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X, \mathcal{E}) \otimes^{L\blacksquare}_K B_{\mathrm{dR}}.
\end{aligned}$$

The first isomorphism follows from the fact that the site $X_{\text{prokét}}$ is coherent, see [DLLZ23b, Proposition 5.1.5]. The second one is the above theorem. The last one follows from the fact that $\otimes^{L\blacksquare}_K$ commutes with filtered colimits, and filtered colimits are exact in $D(\mathrm{Mod}^{\mathrm{cond}}_K)$, as $\mathrm{Mod}^{\mathrm{cond}}_K$ satisfies Grothendieck's axiom (AB5).

Remark 4.10. This theorem allows us to compare the pro-étale cohomology and pro-Kummer étale cohomology of period sheaves. Suppose that X is a smooth rigid analytic varieties over K, $D \subset X$ is a strictly normal crossing divisor, and $U = X - D$. Endow X with the log-structure coming from D. Then the canonical morphism

$$\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathbb{M}) \xrightarrow{\simeq} \mathrm{R}\Gamma_{\underline{\text{proét}}}(U, \mathbb{M})$$

is a (non filtered) quasi-isomorphism in $D(\mathrm{Mod}^{\mathrm{cond}}_K)$. To see this, we use Theorem 4.6, [Bos23a, Theorem 6.5] and Theorem 2.8 (which deduces an isomorphism $\mathrm{logdR}^{\mathcal{E}}_X \simeq \mathrm{dR}^{\mathcal{E}}_U$).

Remark 4.11. One can define filtered $\mathcal{O}_{X_{\text{két}}}$-modules with integrable connections over $X_{\text{két}}$ in the same way, leading to analogous results. In fact, by [DLLZ23b, Definition 4.3.6], a coherent $\mathcal{O}_{X_{\text{két}}}$-module is locally the inverse image of a coherent sheaf on the analytic site of X.

Remark 4.12. If X is a log smooth, proper rigid analytic varieties over K, $(\mathcal{E}, \nabla, \mathrm{Fil}^{\bullet})$ be a filtered $\mathcal{O}_X$-module with integrable log connection, then by Lemma 2.2 and the finiteness of coherent sheaf cohomology on proper rigid analytic varieties, we have $H^i_{\underline{\mathrm{logdR}}}(X,\mathcal{E}) = \underline{H^i_{\mathrm{logdR}}(X,\mathcal{E})}$ for all $i \geq 0$. Therefore

$$H^i_{\underline{\mathrm{logdR}}}(X,\mathcal{E}) \otimes^{\blacksquare}_K B_{\mathrm{dR}} = \underline{H^i_{\mathrm{logdR}}(X,\mathcal{E}) \otimes_K B_{\mathrm{dR}}}.$$

In particular, this recovers the comparison theorem in [DLLZ23a, Theorem 3.2.7], where they generalize [Sch13a, Corollary 1.8].

5. Logarithmic B^+_{dR}-cohomology for rigid analytic varieties

In this chapter, we focus on logarithmic rigid analytic varieties over C. As outlined in the introduction, establishing a meaningful log de Rham-étale comparison theorem requires developing a B^+_{dR}-cohomology theory for such varieties. To achieve this, we adopt the approaches presented in [BMS18] and [Bos23b].

This chapter is dedicated to proving Theorem 1.4, which is essential in formulating the semistable conjecture for the geometric case as stated in [Sha25a]. To maintain focus, we limit our exploration of logarithmic B^+_{dR}-cohomology to the essentials. A more comprehensive study of logarithmic B^+_{dR}-cohomology, along with logarithmic B-cohomology, will be presented in a subsequent article.

5.1. **Preliminary: The $L\eta$ functor.** We review a construction introduced in [BO78] and generalized in [BMS18], which is essential for the construction of logarithmic B^+_{dR}-cohomology.

Let $(T,\mathcal{O}_T)$ be a ringed topos. Denote by $D(\mathcal{O}_T)$ (resp. $K(\mathcal{O}_T)$) the derived category (resp. homotopy category) of $\mathcal{O}_T$-modules. Let $(f) \subset \mathcal{O}_T$ be an invertible ideal sheaf.

Definition 5.1. [BO78, Definition 8.6] Let $M^\bullet \in K(\mathcal{O}_T)$ be an f-torsion-free complex of $\mathcal{O}_T$-modules. Define a new f-torsion-free complex $\eta_f M^\bullet \in K(\mathcal{O}_T)$ as

$$\eta_f(M^\bullet)^i := \{x \in f^i M^i | dx \in f^{i+1}M^{i+1}\}$$

for all $i \in \mathbb{Z}$.

By [BMS18, Remark 6.3], the definition of η_f depends only on the ideal sheaf (f), and it is independent of the generator of (f). According to [BMS18, Corollary 6.5], η_f factors canonically over a (not triangulated!) functor

$$L\eta_f : D(\mathcal{O}_T) \to D(\mathcal{O}_T)$$

on derived categories.

We now define a filtration on $L\eta_f M$. We refer the reader to [BMS19, Chapter 5] for notations and basic proportion on filtered derived ∞-category of $\mathcal{O}_T$-modules. Recall that the filtered derived ∞-category of $\mathcal{O}_T$-modules is defined to be

$$DF(\mathcal{O}_T) := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, D(\mathcal{O}_T)).$$

Let $F \in DF(\mathcal{O}_T)$, we denote by

$$\mathrm{gr}^i(F) := F(i)/F(i+1)$$

the i-th graded piece of F.

Definition 5.2. Let $M \in D(\mathcal{O}_T)$, we define $\mathrm{Fil}^\star L\eta_f M$ a filtration on $L\eta_f M$ as follows: choose a representation $M^\bullet \in K(\mathcal{O}_T)$ of M such that each M^i is f-torsion-free. Then the i-th level of $\mathrm{Fil}^\star L\eta_f M$ is given by

$$\mathrm{Fil}^i L\eta_f M := f^i M^\bullet \cap \eta_f M^\bullet.$$

The well-definedness of this filtration follows from Proposition 5.5.

To understand the filtration on $L\eta_f M$, we recall the following definition introduced in [BMS19], an elaboration of [Bei87, Appendix A].

Definition 5.3. [BMS19, Definition 5.3] Let $DF^{\leq 0}(\mathcal{O}_T) \subset DF(\mathcal{O}_T)$ be the full subcategory consisting of objects F such that $\mathrm{gr}^i(F) \in D^{\leq i}(\mathcal{O}_T)$ for all i. Dually, $DF^{\geq 0}(\mathcal{O}_T) \subset DF(\mathcal{O}_T)$ is the full subcategory consisting of objects F such that $F(i) \in D^{\geq i}(\mathcal{O}_T)$ for all i. We call the pair $(DF^{\leq 0}(\mathcal{O}_T), DF^{\geq 0}(\mathcal{O}_T))$ the Beilinson t-structure on $DF(\mathcal{O}_T)$.

This definition is justified by Theorem 5.4 below.

Theorem 5.4. [BMS19, Theorem 5.4] *(1) The Beilinson t-structure $(DF^{\leq 0}(\mathcal{O}_T), DF^{\geq 0}(\mathcal{O}_T))$ is a t-structure on $DF(\mathcal{O}_T)$.*

(2) Let $\tau_B^{\leq 0}$ denote the connective cover functor associated with the t-structure from (1). Then, there exists a natural isomorphism

$$\mathrm{gr}^i \circ \tau_B^{\leq 0}(-) \simeq \tau^{\leq i} \circ \mathrm{gr}^i(-).$$

(3) The heart of the Beilinson t-structure, defined as

$$DF(\mathcal{O}_T)^{\heartsuit} := DF^{\leq 0}(\mathcal{O}_T) \cap DF^{\geq 0}(\mathcal{O}_T),$$

is equivalent to the abelian category $\mathrm{Ch}(\mathcal{O}_T)$ of chain complexes of $\mathcal{O}_T$-modules. This equivalence can be described as follows: for $F \in DF(\mathcal{O}_T)$, its 0-th cohomology $H_B^0(F)$ in the Beilinson t-structure corresponds to the chain complex $(H^\bullet(\mathrm{gr}^\bullet(F)), d)$, where d is the boundary map induced by the exact triangle

$$\mathrm{gr}^{i+1}(F) = F(i+1)/F(i+2) \to F(i)/F(i+2) \to F(i)/F(i+1) = \mathrm{gr}^i(F).$$

We cite the following proposition of [BMS19], which describes the structure of the filtration of the décalage functor.

Proposition 5.5. [BMS19, Proposition 5.8] *Fix $M \in D(\mathcal{O}_T)$. Let $f^\star \otimes M$ be the f-adic filtration on M, i.e., the i-th level of the filtration of K is $f^i \otimes M$ with obvious maps. Then $L\eta_f M$ is filtered isomorphic with $\tau_B^{\leq 0}(f^\star \otimes M)$ in $DF(\mathcal{O}_T)$.*

We list some results of $L\eta_f$, which will be used in the rest of this chapter.

Proposition 5.6. *(1)*[BMS18, Lemma 6.10] *If $M \in D^{\geq 0}(\mathcal{O}_T)$ such that $H^0(M)[f] = 0$. Then there is a canonical map $\mathrm{Fil}^\star L\eta_f M \to f^\star \otimes M$ in $DF(\mathcal{O}_T)$.*

(2)[BMS18, Proposition 6.12] *For any $M \in D(\mathcal{O}_T)$, we construct a complex $H^\bullet(M/(f))$ with terms*

$$H^i(M/(f)) := H^i(M/^L(f)) \otimes_{\mathcal{O}_T} (f^i),$$

and differential induced by the boundary map corresponding to the short exact sequence

$$0 \to (f)/(f^2) \to \mathcal{O}_T/(f^2) \to \mathcal{O}_T/(f) \to 0.$$

Then there is a natural quasi-isomorphism

$$L\eta_f M \otimes^L_{\mathcal{O}_T} \mathcal{O}_T/(f) \xrightarrow{\simeq} H^\bullet(M/(f)).$$

(3)[BMS18, Lemma 6.14] *Let $u : (T, \mathcal{O}_T) \to (T', \mathcal{O}_{T'})$ be a flat map of ringed topoi. Let $(f) \subset \mathcal{O}_T$ be an invertible ideal sheaf with $(f') := u^*(f) \subset \mathcal{O}_{T'}$, which is still invertible. Then the diagram*

$$\begin{array}{ccc} D(\mathcal{O}_T) & \xrightarrow{u^*} & D(\mathcal{O}_{T'}) \\ \downarrow{\scriptstyle L\eta_f} & & \downarrow{\scriptstyle L\eta_{f'}} \\ D(\mathcal{O}_T) & \xrightarrow{u^*} & D(\mathcal{O}_{T'}) \end{array}$$

commutes, i.e., there is a natural quasi-isomorphism $L\eta_{f'} u^* M \to u^* L\eta_f M$ in $D(\mathcal{O}_{T'})$ for all $M \in D(\mathcal{O}_T)$.

Proof. We only need to check the compatibility for filtration of (1), which follows from the construction in [BMS18, Lemma 6.10]: choosing a representation $M^\bullet$ of M, once we get a map $L\eta_f M \to \otimes M$, we can restrict it to $\mathrm{Fil}^\star L\eta_f M = f^\star M^\bullet \cap \eta_f M^\bullet \to f^\star \otimes M$. □

5.2. **Logarithmic B_{dR}^+-cohomology.** Similar to [Bos23b], we can define the logarithmic B_{dR}^+-cohomology by using the $L\eta$ functor.

Definition 5.7. Let X be a log smooth rigid analytic varieties over C. We denote by $\lambda : X_{\text{prokét}} \to X_{\text{ét,cond}}$ the natural morphism of sites. We define the logarithmic B_{dR}^+-cohomology of X as the complex in $D(\mathrm{Mod}^{\mathrm{solid}}_{B_{\mathrm{dR}}^+})$:

$$\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) := \mathrm{R}\Gamma_{\text{ét,cond}}(X, L\eta_t R\lambda_* \mathbb{B}_{\mathrm{dR}}^+).$$

We endow $\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X)$ with the filtration given by Definition 5.2.

Remark 5.8. Similar to [Guo21], by introducing the logarithmic infinitesimal sites, one is able to define $\mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X/B_{\mathrm{dR}}^+)$, and there should be a canonical filtered quasi-isomorphism:

$$\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) \simeq \mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X/B_{\mathrm{dR}}^+).$$

Proposition 5.9. *Let X be a log smooth rigid analytic variety of dimension d over C. Then $R^i\lambda_* \mathbb{B}_{\mathrm{dR}}^+ = 0$ for all $i > d$.*

Proof. This is an analogy of [Bos23b, Proposition 2.40]. Since the problem is local on $X_{\text{ét,cond}}$ (as both sides satisfy analytic descent), following the notations in Section 3, we may assume $X = \mathrm{Spa}(R, R^+)$ is an affinoid log rigid space over C with a strictly étale morphism $X \to \mathbb{E}_C$, which can be written as a composite of rational embeddings and finite étale maps. By pulling back $\widehat{\widetilde{\mathbb{E}}}_C \to \mathbb{E}_C$ to X we get

$$\widetilde{X} := X \times_{\mathbb{E}_C} \widehat{\widetilde{\mathbb{E}}}_C = \mathrm{Spa}(R_\infty, R_\infty^+),$$

a Galois pro-finite Kummer étale cover with Galois group Γ. The Cartan–Leray spectral sequence gives an quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{cond}}(\Gamma, H^0(\widetilde{X}, \mathbb{B}_{\mathrm{dR}}^+)) \xrightarrow{\simeq} \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathbb{B}_{\mathrm{dR}}^+),$$

which concludes the proof since $\Gamma \simeq \widehat{\mathbb{Z}}(1)^d$ has cohomological dimension d by [Bos23a, B.3] (which is also true for $\Gamma \simeq \widehat{\mathbb{Z}}(1)^d$, by applying [Bos23a, B.4] with $F = \Gamma$ and $R = \widehat{Z}$). □

This proposition allows us to deduce the following two propositions.

Proposition 5.10. *The natural map*

$$L\eta_t R\lambda_* \mathbb{B}_{\mathrm{dR}}^+ \to R\varprojlim_m L\eta_t R\lambda_*(\mathbb{B}_{\mathrm{dR}}^+/t^m)$$

is a quasi-isomorphism compatible with filtrations induced by $L\eta_t$.

Proof. The proof is similar to [Bra18, Lemma 4.3] or [Bos23b, Lemma 2.41]. □

Proposition 5.11. *If X is an affinoid log smooth rigid analytic variety over C, the natural morphism*

$$L\eta_t \, \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{R}\Gamma_{\text{ét,cond}}(X, L\eta_t R\lambda_* \mathbb{B}_{\mathrm{dR}}^+) = \mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X)$$

in $D(\mathrm{Mod}^{\mathrm{solid}}_{B_{\mathrm{dR}}^+})$ is a filtered quasi-isomorphism, where filtrations on both sides are given by the filtration of $L\eta_t$ defined in Definition 5.2.

Proof. The proof is similar to [Bra18, Proposition 3.11] or [Bos23b, Proposition 2.42]. □

Theorem 5.12. *Let X be a log smooth rigid analytic varieties over K then we have a natural isomorphism in $D(\mathrm{Mod}^{\mathrm{solid}}_K)$:*

$$\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X_C) \xrightarrow{\simeq} \mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X) \otimes_K^{L\blacksquare} \underline{B_{\mathrm{dR}}^+},$$

compatible with filtration and G_K-action.

Proof. Since the problem is local (as both sides satisfy analytic descent), we may assume that X is affinoid. In this case, recall that $\lambda : X_{\text{prokét}} \to X_{C,\text{ét,cond}}$, we have

$$\begin{aligned}
\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X_C) &\simeq L\eta_t \, \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X_C, \mathbb{B}_{\mathrm{dR}}^+) \\
&\simeq L\eta_t \, \mathrm{R}\Gamma_{\text{ét,cond}}(X_C, (R\lambda_* \mathbb{B}_{\mathrm{dR}}^+)^{\blacktriangledown}) \\
&\simeq L\eta_t \, \mathrm{R}\Gamma_{\text{ét,cond}}(X_C, \mathrm{Fil}^0(\underline{\mathrm{logdR}}_{X_C} \otimes_K^{\blacksquare} \underline{B_{\mathrm{dR}}})) \\
&\simeq \Big(\underline{\mathcal{O}_{X_C}(X_C)} \xrightarrow{\nabla} \underline{\Omega^{1,\log}_{X_C}(X_C)} \xrightarrow{\nabla} \underline{\Omega^{2,\log}_{X_C}(X_C)} \xrightarrow{\nabla} \cdots \Big) \otimes_K^{L\blacksquare} \underline{B_{\mathrm{dR}}^+} \\
&\simeq \mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X_C) \otimes_K^{L\blacksquare} \underline{B_{\mathrm{dR}}^+},
\end{aligned}$$

where the first quasi-isomorphism follows from Proposition 5.11, the second quasi-isomorphism follows from the same argument as in the proof of Theorem 4.6, the third quasi-isomorphism follows from Proposition 4.8, the forth quasi-isomorphism follows from [Bos23a, Lemma 6.13] and [Bos23a, Theorem 5.20], and the last quasi-isomorphism follows from [Bos23a, Lemma 5.6]. This concludes the proof. □

When X is defined over C, our construction of the B_{dR}^+-cohomology theory indeed provides a deformation of the logarithmic de Rham cohomology. Moreover, we can describe its filtration in a manner analogous to [CN25, Proposition 3.13] in the rigid analytic setting, as follows.

Theorem 5.13. *Let X be a log smooth rigid analytic varieties over C.*

(1) We have a natural isomorphism in $D(\mathrm{Mod}^{\mathrm{solid}}_C)$:

$$\theta : \mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) \otimes^{L\blacksquare}_{B_{\mathrm{dR}}^+} C \xrightarrow{\simeq} \mathrm{R}\Gamma_{\underline{\mathrm{logdR}}}(X).$$

(2) More generally, for $r \geq 0$, we have a natural distinguished triangle in $D(\mathrm{Mod}_C^{\mathrm{solid}})$:

$$\mathrm{Fil}^{r-1} \mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) \xrightarrow{t} \mathrm{Fil}^r \mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) \xrightarrow{\theta} \mathrm{Fil}^r \mathrm{R}\Gamma_{\mathrm{logdR}}(X).$$

(3) For $r \geq 0$, we have a natural distinguished triangle in $D(\mathrm{Mod}_C^{\mathrm{solid}})$:

$$\mathrm{Fil}^{r+1} \mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) \to \mathrm{Fil}^r \mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) \to \mathrm{R}\Gamma(X, \tau^{\leq r} R\lambda_* \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}(r)).$$

Proof. We have

$$\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) \otimes_{B_{\mathrm{dR}}^+}^{L\blacksquare} C \simeq \mathrm{R}\Gamma_{\text{ét,cond}}(X, L\eta_t R\lambda_* \mathbb{B}_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+}^{L\blacksquare} (B_{\mathrm{dR}}^+/t)).$$

We are reduced to showing

$$\left(L\eta_t R\lambda_* \mathbb{B}_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+}^{L\blacksquare} (B_{\mathrm{dR}}^+/t)\right)^{\blacktriangledown} \simeq \underline{\Omega_X^{\bullet,\log}}.$$

By Proposition 5.6 (2), we have

$$\begin{aligned}
\left(L\eta_t R\lambda_* \mathbb{B}_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+}^{L\blacksquare} (B_{\mathrm{dR}}^+/t)\right)^{\blacktriangledown} &\simeq H^\bullet((R\lambda_* \mathbb{B}_{\mathrm{dR}}^+)/t)^{\blacktriangledown} \simeq H^\bullet(R\lambda_*(\mathbb{B}_{\mathrm{dR}}^+/t))^{\blacktriangledown} \simeq H^\bullet(R\lambda_* \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}})^{\blacktriangledown} \\
&\simeq \left(R^\bullet \lambda_* \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}\right)^{\blacktriangledown} \otimes_{B_{\mathrm{dR}}^+} \mathrm{Fil}^\bullet B_{\mathrm{dR}}^+ \\
&\simeq \underline{\Omega_X^{\bullet,\log}(-\bullet)} \otimes_C C(\bullet) \\
&\simeq \underline{\Omega_X^{\bullet,\log}},
\end{aligned}$$

where the second-to-last isomorphism follows from the proposition below: to see that the Bockstein-type differential coincides with the usual log-differential on $\Omega_X^{\bullet,\log}$ (this is a much easier version of [BMS18, Theorem 8.3], see also [Bha18, Proposition 7.9]), by Proposition 5.15 we may reduce to checking that the Bockstein-type map

$$\mathcal{O}_X \simeq R^0\lambda_* \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}} \to R^1\lambda_* \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}(1) \simeq \Omega_X^{1,\log}$$

is the usual log-differential. We may work locally, translating the elements in $R^1\lambda_* \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}(1)$ to cocycle conditions in group cohomology, where the same calculation of the proof of Proposition 5.22 applies. This concludes the proof of (1). Then (2) follows from [Wu24, Proposition 3.4], and (3) follows from [Wu24, Lemma 3.3]. □

Remark 5.14. We will see that when X descends to a log smooth X_0 over a discretely valued field K such that $X = X_0 \times_K C$, we have a natural quasi-isomorphism

$$\tau^{\leq r} R\lambda_* \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}(r) \simeq \bigoplus_{i \leq r} \Omega_X^{i,\log}(r-i)[-i],$$

and in general, a choice of a log smooth B_{dR}^+/t^2-lift $\mathbb{X}$ of X via the map $B_{\mathrm{dR}}^+/t^2 \to C$ gives such a quasi-isomorphism.

Proposition 5.15. *Let X be a log smooth rigid analytic varieties over C, then we have a natural isomorphism*

$$\left(R^i \lambda_* \widehat{\mathcal{O}}_{X_{\mathrm{prok\acute{e}t}}}\right)^{\blacktriangledown} \simeq \underline{\Omega_X^{i,\log}(-i)}$$

for all $i \geq 0$.

Proof. We can use the same argument as in the proof of [Sch13b, Proposition 3.23]: we first prove that $\mathcal{E} := R^1\lambda_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}$ is a locally free $\mathcal{O}_{X_{\text{ét}}}$-module of rank dim X, such that

$$\bigwedge^j \mathcal{E} \simeq R^j\lambda_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}.$$

This can be checked locally, by following a direct computation using the Koszul complex of Γ (as in the proof of [Sch13a, Lemma 5.5], which also requires the use of [DLLZ23b, Lemma 6.1.9]). Then, the following lemma, which is a log-version of [Sch13b, Lemma 3.24], concludes the proof. □

Lemma 5.16. *Consider the exact sequence (for example, see* [KN99, Proposition 2.3]*)*

$$0 \to \mathbb{Z}_p(1) \to \varprojlim_{\times p} \nu^{-1}\mathcal{M}_X^{gp} \to \nu^{-1}\mathcal{M}_X^{gp} \to 0$$

on $X_{\text{prokét}}$. Here ν is the natural morphism of sites $\nu : X_{\text{prokét}} \to X_{\text{ét}}$. It induces a boundary map $\mathcal{M}_X^{gp} = \nu_*\nu^{-1}\mathcal{M}_X^{gp} \to R^1\lambda_*\mathbb{Z}_p(1)$. Then there is a unique $\underline{\mathcal{O}_X}$-linear map $\underline{\Omega_X^{\log}} \to R^1\lambda_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}(1)^{\blacktriangledown}$ such that the diagram

$$\begin{tikzcd} \mathcal{M}_X^{gp} \arrow[r] \arrow[d, "\delta"] & R^1\nu_*\mathbb{Z}_p(1) \arrow[d] \\ \underline{\Omega_X^{\log}}(*) \arrow[r] & R^1\lambda_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}(1)^{\blacktriangledown}(*) \end{tikzcd}$$

commutes, where $\delta : \mathcal{M} \to \Omega^{1,\log}$ is the derivation. This map is an isomorphism.

Proof. The proof of [Sch13b, Lemma 3.24] goes through. The point is to reduce the claim to the arithmetic case, i.e. X descends to a log smooth X_0 over a discretely valued field K such that $X = X_0 \times_K C$. See also [LP19, Corollary 6.15] and [LP19, Remark 6.17] in the logarithmic settings. □

Similar to the log de Rham cohomology, we can compare the logarithmic B_{dR}^+-cohomology with the B_{dR}^+-cohomology of the trivial locus when the log structure comes from a strictly normal crossing divisor.

Proposition 5.17. *Let X be a log smooth rigid analytic varieties over C with log-structure given by a strictly normal crossing divisor $D \subset X$. Denote by $U := X - D$. Then, we have a natural (non filtered!) isomorphism in $D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{solid}})$:*

$$\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) \simeq \mathrm{R}\Gamma_{\underline{\mathrm{dR}}}(U/B_{\mathrm{dR}}^+).$$

Proof. Since both sides satisfy analytic descent, the problem is local, and we may assume X can be descent to an log variety X_0 over a discrete valued field K by the lemma below. Then the claim follows from Theorem 5.12, [Bos23b, Theorem 5.1] and Theorem 2.8. □

Lemma 5.18. *If X is a log smooth rigid analytic varieties over C of dimension d with log-structure giving by a strictly normal crossing divisor $D \subset X$. Then étale locally X has a basis of rigid analytic varieties consisting of objects of the form $\mathrm{Sp}(A_C\langle x_1, ..., x_d\rangle)$ with log structure given by the normal crossing divisor $x_1 \cdots x_d = 0$, where $\mathrm{Sp}(A)$ is a smooth rigid analytic variety over K for some discretely valued field K in C.*

Proof. By the existence of tubular neighborhood, [Kie67, Theorem 1.18], locally X can be written as $V := \mathrm{Sp}(R\langle x_1, ..., x_d\rangle)$, and D is defined by the equation $x_1 \cdots x_d = 0$. Then the claim follows from Temkin's alteration theorem [Tem17, Theorem 3.3.1]. □

5.3. **Comparison with the pro-Kummer étale cohomology.** The following comparison theorem can be easily deduced from our construction for logarithmic B_{dR}^+-cohomology.

Theorem 5.19. *Let X be a log smooth proper rigid analytic variety over C. Then we have a canonical filtered isomorphism in* CondAb :

$$H^i_{\underline{\text{két}}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq H^i_{B_{\mathrm{dR}}^+}(X) \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}.$$

Here, the filtration on $H^i_{B_{\mathrm{dR}}^+}(X)$ is defined by

$$\mathrm{Fil}^\star H^i_{B_{\mathrm{dR}}^+}(X) := \mathrm{Im}(H^i(\mathrm{Fil}^\star \mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X)) \to H^i_{B_{\mathrm{dR}}^+}(X)).$$

Moreover, when X descends to a log smooth X_0 over a discretely valued field K such that $X = X_0 \times_K C$, this isomorphism agrees with the comparison isomorphism

$$H^i_{\underline{\text{két}}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq H^i_{\underline{\mathrm{logdR}}}(X_0) \otimes_K B_{\mathrm{dR}},$$

under the canonical identification in Theorem 5.12.

Proof. We will construct a morphism:

$$\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) \to \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathbb{B}_{\mathrm{dR}}^+) \simeq \mathrm{R}\Gamma_{\underline{\text{két}}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+,$$

where the last isomorphism follows from [DLLZ23a, Lemma 3.6.1] (whose proof also works over C, by the same proof as presented in [Sch13a, Theorem 8.4]). Then we will prove that such a morphism is a quasi-isomorphism after inverting t.

By Proposition 5.6 (1), we have a natural morphism

$$\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X) = \mathrm{R}\Gamma_{\text{ét,cond}}(X, L\eta_t R\lambda_* \mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{R}\Gamma_{\text{ét,cond}}(X, R\lambda_* \mathbb{B}_{\mathrm{dR}}^+) = \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathbb{B}_{\mathrm{dR}}^+),$$

which gives the desired morphism.

To show that such a morphism is a quasi-isomorphism after inverting t, as both sides satisfy analytic descent, it suffices to prove it locally so we can assume that X is affinoid. By Proposition 5.6 (3), considering the base change $B_{\mathrm{dR}}^+ \to B_{\mathrm{dR}}$, picking $\mathcal{O}_{T'} := B_{\mathrm{dR}}^+, \mathcal{O}_T := B_{\mathrm{dR}}$ and $M := \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathbb{B}_{\mathrm{dR}}^+)$, we have a quasi-isomorphism

$$\mathrm{R}\Gamma_{B_{\mathrm{dR}}^+}(X)\left[\frac{1}{t}\right] \simeq L\eta_t\left(\mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathbb{B}_{\mathrm{dR}}^+)\left[\frac{1}{t}\right]\right) \simeq \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathbb{B}_{\mathrm{dR}}),$$

which is compatible with filtration: for any integer r, we have

$$\mathrm{Fil}^r\left(L\eta_t M\left[\frac{1}{t}\right]\right) = \sum_{i+j=r} t^j\left(t^i M \cap L\eta_t M\right) \subseteq t^r M = t^r\, \mathrm{R}\Gamma_{\underline{\text{prokét}}}(X, \mathbb{B}_{\mathrm{dR}}^+).$$

This concludes the proof.

When X descends to X_0 over a discretely valued field K, the compatibility stated in the theorem is clear due to Theorem 5.12. □

5.4. **Degeneration of (log) Hodge–Tate spectral sequence.** This section is devoted to proving the following theorem, as promised in the introduction. For convenience, we will only work with usual cohomology groups in this section. However, clearly all results can be extended to condensed cohomology group without any difficulty.

Recall that we denote by ν the natural morphism of sites $\nu : X_{\text{prokét}} \to X_{\text{ét}}$.

Theorem 5.20. *Let X be a proper log smooth rigid analytic variety over C of dimension d, then*
(i) the Hodge–log de Rham spectral sequence

$$E_1^{ij} := H^j(X, \Omega_X^{\log,i}) \Rightarrow H^{i+j}_{\mathrm{logdR}}(X)$$

degenerates at E_1.
(ii) the Hodge–Tate spectral sequence

$$E_2^{ij} := H^j(X, \Omega_X^{\log,i})(-j) \Rightarrow H^{i+j}_{\text{két}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$$

degenerates at E_2.
Moreover, $H^i_{B^+_{\mathrm{dR}}}(X/B^+_{\mathrm{dR}})$ is a finite free B^+_{dR}-module.

The key step in the proof is to demonstrate the existence of a splitting

$$\bigoplus_{i\geq 0} \Omega_X^{i,\log}(-i)[-i] \xrightarrow{\simeq} R\nu_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}.$$

For rigid analytic varieties, this is shown in [Guo23, Proposition 7.2.5] by choosing a smooth B^+_{dR}/t^2-lifting $\mathbb{X}$ of X. Such a lifting always exists when X is proper. Additionally, the splitting depends on the choice of lifting functorially.

We now turn our attention to the logarithmic case, discussing conditions under which such a splitting exists.

5.4.1. *$\mathbb{B}^+_{\mathrm{dR}}/t^2$-liftings of X.* As demonstrated in [BMS18] or [Guo23], a key step in the proof of the theorem is to establish the existence of a $\mathbb{B}^+_{\mathrm{dR}}/t^2$-lifting for proper X. We will show that the same holds for log smooth and proper adic space.

Proposition 5.21. *Let X be a log smooth rigid analytic varieties over C. Then X admits a log smooth B^+_{dR}/t^2-lift $\mathbb{X}$ via the map $B^+_{\mathrm{dR}}/t^2 \to C$ in the following cases:*
(1) X descends to X_0 over a discretely valued field K;
(2) X is proper.

Proof. (1) is clear as the base change of X_0 along $K \to B^+_{\mathrm{dR}}/t^2$ gives a desired lift.

For (2), denote by $U := X_{\mathrm{tr}}$ the trivial locus of X (i.e. largest open where the induced log structure is trivial), and $D := X - U$ the closed complement of U. Since X is log smooth, according to [Ogu18, Corollary 1.9.5], we can identify the log structure of X with the compactifying structure induced by U. By Raynaud's theory on formal and rigid geometry, and Temkin's results on analytic spaces in [Tem00], there exists a closed immersion $\mathfrak{D} \hookrightarrow \mathfrak{X}$ of proper flat formal schemes over $\mathrm{Spf}(\mathcal{O}_C)$ such that the closed immersion $D \hookrightarrow X$ is the rigid generic fiber of the morphism $i : \mathfrak{D} \hookrightarrow \mathfrak{X}$.

Choose an inclusion of fields $\breve{F} \to C$. We will show that there exists a smooth rigid space $\mathcal{S}$ over $\breve{F}$, such that the closed immersion $D \to X$ can be lifted to a morphism between flat $\breve{F}$-rigid spaces $D_A \to X_A$ over $\mathcal{S}$ with a map $\mathrm{Spa}(C, \mathcal{O}_C) \to \mathcal{S}$. Then since $B^+_{\mathrm{dR}}/t^2 \to C$ has a natural $\breve{F}$-structure, by formal smoothness the map $\mathrm{Spa}(C,\mathcal{O}_C)\to\mathcal{S}$ can be lifted to a map $\mathrm{Spa}(B_{\mathrm{dR}}^+/\xi^2, A_{\mathrm{inf}}/\xi^2)\to\mathcal{S}$. This gives a B_{dR}^+/t^2-lifting $\mathbb{D}\to\mathbb{X}$ of $D\to X$. The open complement $\mathbb{U}$ of $\mathbb{D}$ then induces the compactifying log structure on $\mathbb{X}$, which also gives a B_{dR}^+/t^2-lifting of the log adic space X by [Ogu18, Proposition 1.6.2]. Moreover, $\mathbb{X}$ is log smooth over B_{dR}^+/t^2 by [Ogu18, Proposition 4.2.4]. This will finish the proof.

Denote by $\mathscr{C}_{\mathcal{O}_{\breve{F}}}$ the category of complete local artin $\mathcal{O}_{\breve{F}}$-algebras with residue field $\overline{k}$. Consider the deformation functor

$$\mathbf{D}:\mathscr{C}_{\mathcal{O}_{\breve{F}}}\to\mathbf{Set}$$

defined by sending $A\in\mathscr{C}_{\mathcal{O}_{\breve{F}}}$ to the isomorphism class of flat liftings of $i_s:\mathfrak{D}_s\to\mathfrak{X}_s$ on A. According to [sta, 0E3S], the deformation functor $\mathbf{D}$ has a versal deformation, i.e., a complete noetherian local $\mathcal{O}_{\breve{F}}$-algebra A with the residue field $\overline{k}$, a morphism between proper flat formal scheme $i_A:\mathfrak{D}_A\to\mathfrak{X}_A$ over A where A is topologized by powers of its maximal ideal, deforming i_s such that the induced classifying map

$$h_A:\mathrm{Hom}_{\mathcal{O}_{\breve{F}}}(A,-)\to\mathbf{D}$$

is formally smooth. According to the proof of [BMS18, Proposition 13.15], we can extend h_A and $\mathbf{D}$ to the ind-completion of $\mathscr{C}_{\mathcal{O}_{\breve{F}}}$. This category includes $\mathcal{O}_C/\omega^k$ for all $k\geq 1$. Denote by $i_k:\mathfrak{D}_k\to\mathfrak{X}_k$ the reduction of $i:\mathfrak{D}\to\mathfrak{X}$ module ω^k. Since we have a canonical map $\eta_0:A\to\overline{k}$, and an isomorphism $\psi_0:\eta_0^*i_A\simeq i_s$, applying the formal smoothness of h_A (on the ind-completion of $\mathscr{C}_{\mathcal{O}_{\breve{F}}}$) to the $\mathcal{O}_C/\omega\to\overline{k}$, we can choose a map $\eta_1:A\to\mathcal{O}_C/\omega$ lifting η_0 and an isomorphism $\psi_1:\eta_1^*i_A\simeq i_1$ of morphism beteween $\mathcal{O}_C/\omega$-schemes lifting ψ_0. We can then construct ψ_n and η_n inductively, taking the inverse limit, and we can show that there exists a map $\eta:A\to\mathcal{O}_C$ of $\mathcal{O}_{\breve{F}}$-algebra, and an isomorphism $\eta^*i_A\simeq i$ as morphism between formal $\mathcal{O}_C$-schemes.

Finally, after inverting p we have a morphism of flat $\breve{F}$-rigid spaces $D_A\to X_A$ over $\mathcal{S}:=\mathrm{Spa}(A,A[1/p])$. We can shrink $\mathcal{S}$ to a suitable small locally closed subset, such that $\mathcal{S}$ is smooth over $\breve{F}$. This concludes the proof. □

5.4.2. *Splitting of the (log) Hodge–Tate map.* We will prove the following proposition.

Proposition 5.22. *Let X be a log smooth rigid analytic varieties over C. Then a choice of log smooth B_{dR}^+/t^2-lift $\mathbb{X}$ of X gives a natural splitting of the (log) Hodge-Tate map*

$$\mathrm{HT}:H^1_{\text{prokét}}(X,\widehat{\mathcal{O}}_{X_{\text{prokét}}})\to H^0(X,\Omega_X^{\log}(-1)).$$

Here the map HT *is induced by the spectral sequence*

$$E_2^{ij}:=H^j_{\text{ét}}(X,R^i\nu_*\widehat{\mathcal{O}}_{X_{\text{prokét}}})\Rightarrow H^{i+j}_{\text{prokét}}(X,\widehat{\mathcal{O}}_{X_{\text{prokét}}}).$$

We prove this proposition in the fashion of [Heu25, Proposition 2.15]. Via the homeomorphism $|\mathbb{X}|=|X|$, we may regard $\mathcal{O}_{\mathbb{X}}$ as the sheaf on $X_{\text{ét}}$. Define

$$L_{\mathbb{X}}:=\left\{\begin{array}{l|l}\text{homomorphisms } \widetilde{\varphi} \text{ of sheaves of } B_{\mathrm{dR}}^+/t^2\text{-algebras on } X_{\text{prokét}} \text{ such that the right square is commutative:} & \begin{array}{ccc}\nu^{-1}\mathcal{O}_{\mathbb{X}} & \xrightarrow{\widetilde{\varphi}} & \mathbb{B}_{\mathrm{dR}}^+/t^2\\ \downarrow & & \downarrow\\ \nu^{-1}\mathcal{O}_X & \longrightarrow & \widehat{\mathcal{O}}_{X_{\text{prokét}}}\end{array}\end{array}\right\}.$$

Note that $L_{\mathbb{X}}$ is a sheaf on $X_{\text{prokét}}$ as a subsheaf of $\mathcal{H}om(\lambda^{-1}\mathcal{O}_{\mathbb{X}},\mathbb{B}_{\mathrm{dR}}^+/t^2)$. We have

Lemma 5.23. *$L_{\mathbb{X}}$ is a pro-Kummer étale torsor under $\nu^*\Omega_X^{\log}(-1)^\vee$.*

Proof. Similar to the proof of [Heu25, Lemma 2.13] by using the theory of infinitesimal liftings for log smooth morphism, see [Kat89, Proposition 3.14]. □

Proof of Propostion 5.22. For $\theta \in H^0(X, \Omega_X^{\log}(-1))$, define a map

$$s_{\mathbb{X}} : H^0(X, \Omega_X^{\log}(-1)) \to H^1_{\text{prokét}}(X, \widehat{\mathcal{O}}_{X_{\text{prokét}}})$$

by

$$s_{\mathbb{X}} : \theta \mapsto L_{\mathbb{X},\theta} := L_{\mathbb{X}} \times^{\nu^*\Omega_X^{\log}(-1)^\vee} \nu^*\mathcal{O}_X$$

via identifying $\Omega_X^{\log}(-1)^\vee \to \mathcal{O}_X$, which is induced from the dual of the canonical map $\mathcal{O}_X \to \Omega_X^{\log}(-1)$. We need to check that $\mathrm{HT}(s_{\mathbb{X}}(\theta)) = \theta$.

The problem is local, so we may assume $X = \mathrm{Spa}(R, R^+)$ is an affinoid log rigid space over C with a strictly étale morphism $X \to \mathbb{E} = \mathrm{Spa}(C\langle P\rangle, \mathcal{O}_C\langle P\rangle)$, which can be written as a composite of rational embeddings and finite étale maps. Let T^a be the induced coordinate on X, for any $a \in P$. Let $\widetilde{X} = \mathrm{Spa}(R_\infty, R_\infty^+) \to X$ be the Galois pro-finite Kummer étale cover with Galois group Γ as before. Choose a $\mathbb{Z}$-basis $\{a_1, ..., a_d\}$ of P^{gp}, where $d = \dim X$. Then $\delta(a_1), ..., \delta(a_d)$ form a basis of $\Omega_R^{\log}$, where $\delta : P \to \Omega_R^{\log}$ is the derivation. Denote by $\partial_1, ..., \partial_d$ its dual basis in $\Omega_R^{\log,\vee}$.

For any lift $\mathbb{X} = \mathrm{Spa}(R', R'^+)$, by formal log smoothness there exists a lift of the map $C\langle P\rangle \to R$ induced by $X \to \mathbb{E}$ to an étale morphism

$$B_{\mathrm{dR}}^+/t^2\langle P\rangle \to R'$$

of B_{dR}^+/t^2-algebras. Here we view B_{dR}^+/t^2 as a Tate $\mathbb{Q}_p$-algebra with a ring of definition A_{inf}/ξ^2, and the log structure on $P \to B_{\mathrm{dR}}^+/t^2\langle P\rangle$ is given by $a \mapsto T^a$. Any choice of such a lift induces a section of $L_{\mathbb{X}}(\widetilde{X})$ as follows: the morphism

$$B_{\mathrm{dR}}^+/t^2\langle P\rangle \to \mathbb{B}_{\mathrm{dR}}^+/t^2(\widetilde{X}) : T^a \to [T^{a/p^\infty}] \text{ for any } a \in P$$

extends by formal log étaleness to a unique map $R' \to \mathbb{B}_{\mathrm{dR}}^+/t^2(\widetilde{X})$ lifting the map $R \to R_\infty$. The following lemma describe the Γ-action on $L_{\mathbb{X}}(\widetilde{X})$.

Lemma 5.24. *Write $a_i = a_i^+ - a_i^-$ with $a_i^+, a_i^- \in P$ for $i = 1, 2, ..., d$. Let $c_i^+, c_i^- : \Gamma \to \mathbb{Z}_p(1) = \varprojlim_n \boldsymbol{\mu}_{p^n}$ be the maps such that for any $\gamma \in \Gamma$, we have*

$$\gamma[T^{a_i^+/p^\infty}] = [c_i^+(\gamma)][T^{a_i^+/p^\infty}], \gamma[T^{a_i^-/p^\infty}] = [c_i^-(\gamma)][T^{a_i^-/p^\infty}].$$

Then under the identification $L_{\mathbb{X}}(\widetilde{X}) = \Omega_R^{\log}(-1)^\vee \otimes_R R_\infty$ induced by $R' \to B_{\mathrm{dR}}^+/t^2\langle P_{\mathbb{Q}>0}\rangle$, the Γ-action on the right can be described the continuous 1-cocycle

$$\Gamma \to \Omega_R^{\log,\vee}(1), \gamma \mapsto \sum_{i=1}^d \frac{c_i^+(\gamma)}{c_i^-(\gamma)}\partial_i,$$

via the natural identification $H^1_{\text{prokét}}(X, \nu^*\Omega_X^{\log}(-1)^\vee) \simeq H^1(\Gamma, \Omega_R^{\log,\vee}(1))$.

Proof. The proof is similar to [Heu25, Lemma 2.16], let $R_\infty\{1\} := \ker(\mathbb{B}_{\mathrm{dR}}^+/t^2(R_\infty) \to R_\infty)$. For any $\Sigma x_i\partial_i \in \mathrm{Hom}(\Omega_R^{\log}, R_\infty\{1\})$, denote by $x_i^+ := x_i\partial_i(\delta(a_i^+)), x_i^- := x_i\partial_i(\delta(a_i^-))$ for all i, then the corresponding element in $L_{\mathbb{X}}(R_\infty) \simeq \Omega_R^{\log}(-1)^\vee \otimes_R R_\infty$ is given by the morphism induced by sending $T^{a_i^+} \mapsto [T^{a_i^+/p^\infty}] + x_i^+$ and $T^{a_i^-} \mapsto [T^{a_i^-/p^\infty}] + x_i^-$. Then to describe the γ-action, we compute:

$$\gamma([T^{a_i^+/p^\infty}] + x_i^+) - [T^{a_i^+/p^\infty}] = ([c_i^+(\gamma)] - 1)[T^{a_i^+/p^\infty}] + \gamma x_i^+,$$
$$\gamma([T^{a_i^-/p^\infty}] + x_i^-) - [T^{a_i^-/p^\infty}] = ([c_i^-(\gamma)] - 1)[T^{a_i^-/p^\infty}] + \gamma x_i^-.$$

This concludes the proof, since we can identify $[\epsilon]-1$ by t in B_{dR}/t^2, for $\epsilon = (1, \cdots) \in \mathbb{Z}_p(1)$, which induces a canonical isomorphism $R\{1\} \to R(1)$. □

For any $i = 1, ..., d$, consider the $\widehat{\mathcal{O}}_{X_{\text{prokét}}}(1)$ torsor $L_{\mathbb{X}} \times^{\nu^*\Omega_X^{\log,\vee}(1)} \nu^*\mathcal{O}_X(1)$ induced by $\delta(a_i) : \Omega_X^{\log,\vee} \to \mathcal{O}_X$. We need to check its image under $\mathrm{HT}(1)$ is $\delta(a_i)$. This follows from the commutative diagram in Lemma 5.16: $\delta(a_i)$ maps to $\mathrm{HT}(1)^{-1}\delta(a_i)$ through the map over the lower left corner, and it maps to the class defined by the 1-cocycle $\Gamma \to R(1) : \gamma \to c_i^+(\gamma)/c_i^-(\gamma)$ through the map over the upper right corner, which is exactly the cocycle associated to $L_{\mathbb{X}} \times^{\nu^*\Omega_X^{\log,\vee}(1)} \nu^*\mathcal{O}_X(1)$ by the lemma above. This concludes the proof. □

5.4.3. *Splitting of $R\nu_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}$.* The splitting of the Hodge–Tate map induces a natural decomposition of $R\lambda_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}$ as follows.

Proposition 5.25. *Let X be a log smooth rigid analytic varieties over C. Then a choice of log smooth B_{dR}^+/t^2-lift $\mathbb{X}$ of X gives a natural splitting of $R\nu_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}$ as $\bigoplus_{i\geq 0}\Omega_X^{i,\log}(-i)[-i]$ in the derived category.*

Proof. The splitting of the (log) Hodge–Tate map gives a natural isomorphism

$$\mathcal{O}_X \oplus \Omega_X^{1,\log}(-1)[-1] \xrightarrow{\simeq} \tau^{\leq 1} R\nu_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}.$$

The quotient map

$$\left(\Omega_X^{1,\log}\right)^{\otimes_{\mathcal{O}_X} i} \to \bigwedge^i \Omega_X^{1,\log} = \Omega_X^{i,\log}$$

has a canonical $\mathcal{O}_X$-linear section s_i given by

$$\omega_1 \wedge ... \wedge \omega_i \mapsto \frac{1}{i!}\sum_{\sigma\in S_i} \mathrm{sgn}(\sigma)\omega_{\sigma(1)} \otimes ... \otimes \omega_{\sigma(i)}.$$

Here, we denote by $(-)^{\otimes i}$ the tensor product of i copies of the object. Therefore, we can construct a map

$$\Omega_X^{i,\log}(-i)[-i] \to (\Omega_X^{1,\log}(-1)[-1])^{\otimes_{\mathcal{O}_X} i} \to (R\lambda_*\widehat{\mathcal{O}}_{X_{\text{prokét}}})^{\otimes_{\mathcal{O}_X} i} \to R\nu_*\widehat{\mathcal{O}}_{X_{\text{prokét}}},$$

which induces a natural map

$$\bigoplus_{i\geq 0} \Omega_X^{i,\log}(-i)[-i] \to R\nu_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}.$$

The map constructed above is a quasi-isomorphism by (the construction of) Proposition 5.15. □

5.4.4. *Proof of Theorem 5.20.* With all these preparations, the proof for Theorem 5.20 is then followed by counting dimensions.

We start with (2). The spectral sequence

$$E_2^{ij} := H^j_{\text{ét}}(X, R^i\nu_*\widehat{\mathcal{O}}_{X_{\text{prokét}}}) \Rightarrow H^{i+j}(X_{\text{prokét}}, \widehat{\mathcal{O}}_{X_{\text{prokét}}}) \simeq H^{i+j}_{\text{két}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$$

converges, where the last isomorphism follows from the primitive comparison theorem in [DLLZ23b, Theorem 6.2.1]. The claim then follows from Proposition 5.25 by counting dimensions.

To show (1), we need to show that

$$\sum_{i=1}^{d} \dim_C H^j(X, \Omega_X^{i,\log}) = \dim_C H^{i+j}_{\log\mathrm{dR}}(X).$$

By (2), we have

$$\sum_{i=1}^{d} \dim_C H^j(X, \Omega_X^{i,\log}) = \dim_{\mathbb{Q}_p} H^{i+j}_{\text{két}}(X, \mathbb{Q}_p).$$

On the other hands, Theorem 5.19 shows

$$\begin{aligned}
\dim_{\mathbb{Q}_p} H^{i+j}_{\text{két}}(X, \mathbb{Q}_p) &= \dim_{B_{\mathrm{dR}}} H^{i+j}_{B^+_{\mathrm{dR}}}(X)(*)\left[\frac{1}{t}\right] \\
&= \dim_{B^+_{\mathrm{dR}}} H^{i+j}_{B^+_{\mathrm{dR}}}(X)(*)/\mathit{torsion} \\
&\le \dim_{B^+_{\mathrm{dR}}} H^{i+j}_{B^+_{\mathrm{dR}}}(X)(*)/t \\
&\le \dim_C H^{i+j}_{\log\mathrm{dR}}(X),
\end{aligned}$$

where the second equality holds since $H^i_{B^+_{\mathrm{dR}}}(X)(*)/\mathit{torsion}$ is a finite rank torsion-free module over B^+_{dR} without nontrivial divisible submodule, so it must be free by [Rot60, Theorem 3]. The last inequality follows from Theorem 5.13, which induces a short exact sequence

$$0 \to H^{i+j}_{B^+_{\mathrm{dR}}}(X)(*)/t \to H^{i+j}_{\log\mathrm{dR}}(X) \to H^{i+j+1}_{B^+_{\mathrm{dR}}}(X)(*)[t] \to 0.$$

This proves (1) as all inequalities must be equality.

Finally, the above computation also shows that $H^i_{B^+_{\mathrm{dR}}}(X)$ is a finite free B^+_{dR}-module as it is torsion-free. This finishes the proof.

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